1.051 Structural Engineering Design

Problem Set 2 Solutions

Problem 1

$LL = 1500$ lb/ft
$DL = 300$ lb/ft
$L = 26$ ft
$h = 22$ in
$fy = 60,000$ psi
$fc = 5,000$ psi

(a) Beam design:

Let $b = 12$ in in and $d = h - 2.5 = 19.5$ in

Beam weight: $w = (12)(22)\frac{150}{12^2} = 275$ lb/ft

Design load: $W_u = 1.2(300 + 275) + 1.6(1500) = 3090$ lb/ft

Design moment: $M_u = W_u \frac{L^2}{8} = 3090\frac{26^2}{8} = 261.1$ kips-ft = 3133 kips-in

Minimum and maximum reinforcement ratios:

$$\beta_1 = 0.85 - \frac{5000 - 4000}{1000}0.05 = 0.80$$

balanced ratio;

$$r_b = 0.85 b_1 \frac{f_c'}{f_y}\frac{e_u}{e_u + e_y} = (0.85)(0.80)\frac{5}{60}\frac{0.003}{0.003 + \frac{60}{29000}}$$
$$= 0.0335$$

maximum reinforcement ratio

$$r_{max} = 0.75 r_b = 0.75(0.0335) = 0.0251$$

alternatively $$r_{max} = 0.85 b_1 \frac{f_c'}{f_y}\frac{e_u}{e_u + 0.004} = (0.85)(0.80)\frac{5}{60}\frac{0.003}{0.007} = 0.0243$$

minimum reinforcement ratio:

$$r_{min} = \frac{3\sqrt{f_c'}}{f_y} \;{}^{3}\; \frac{200}{f_y}$$
$$= \frac{3\sqrt{5000}}{60000} \;{}^{3}\; \frac{200}{60000}$$
$$= 0.0035 \;{}^{3}\; 0.0033$$

Write the expression for nominal moment capacity and solve for r

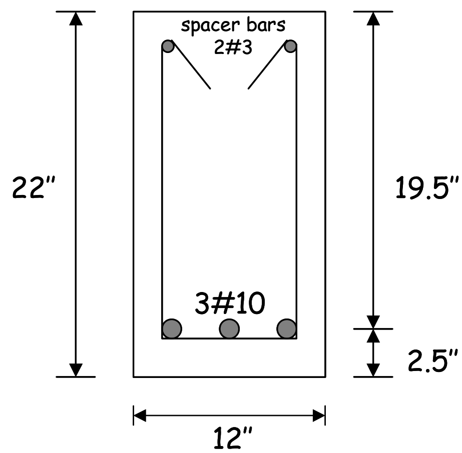
$$M_n = \frac{M_u}{f} = r b d^2 f_y (1 - 0.59 r \frac{f_y}{f_c'})$$

$$= \frac{3077}{0.9} = r(12)(19.5)^2(60)(1 - 0.59 r \frac{60}{5})$$

solve for r Þ $r = 0.014$ $r_{min} < r < r_{max}$ OK

Reinforcement area: $A_s = r bd = 0.014(12)(19.5) = 3.3 \text{ in}^2$

Use 3#10 $A_s = 3(1.27) = 3.81 \text{ in}^2$ OK



Spacing of bars complies with ACI requirements given in Appendix Table A7 in the textbook.

(b) <u>Major assumptions</u>

- Strain compatibility assumption – plane sections remain plane
- Tensile strength of concrete is ignored
- Perfect bond between concrete and steel
- Compressive stress-strain behavior of concrete and tensile stress-strain behavior of steel follows the associated material stress-strain diagrams

If these assumptions did not hold, the analysis of the beam would be highly complex and would not lend itself to a simple analysis by hand calculations.

(c) Formation of the first crack

First crack occurs when the maximum tensile stress in concrete reaches the modulus of rupture f_r of concrete

$f_{ct}^{max} = f_r$ Tensile stress calculated from elastic theory ignoring the steel reinforcement

$$f_{ct} = \frac{My}{I}, \quad f_{ct}^{max} = \frac{M\frac{h}{2}}{I} = \frac{M}{S}, \quad S = \frac{bh^2}{6}$$

$$f_{ct}^{max} = \frac{6M}{bh^2} \quad \text{and} \quad f_r = 7.5\sqrt{f_c'} \quad \text{(Textbook Section 2.9.a)}$$

$$\frac{6M}{bh^2} = 7.5\sqrt{f_c'} \ ® \ M_{cr} = \frac{7.5}{6}bh^2\sqrt{f_c'} = \frac{7.5}{6}(12)(22^2)\sqrt{5000} = 513{,}359 \text{ lb-in}$$

$$M_{cr} = \frac{w_{cr}L^2}{8} = 42780 \text{ lb-ft}$$

$$® \ w_{cr} = \frac{(42780)(8)}{26^2} = 506 \text{ lb/ft}$$

Before crack formation, the neutral axis coincides with the controidal axis of the section, which can be approximated as at h/2 ignoring the reinforcement. After crack formation, the neutral axis moves up since the cracked portion of the concrete no longer contributes to the section stiffness.

(d) Control of cracking

Moment under service loading:

$$M = \frac{w_s L^2}{8} = \frac{(275 + 300 + 1500)(26^2)}{8} = 175{,}337 \text{ lb-ft} = 2104 \text{ kips-in}$$

Under service loading, assume linear stress-strain distribution in concrete and steel.

Elastic modulus of concrete:

$$E_c = 57000\sqrt{f_c'} = 57000\sqrt{5000} = 4{,}030{,}509 \text{ psi} = 4{,}031 \text{ ksi}$$

(Eq. 2.4 in textbook)

modular ratio:

$$n = \frac{E_s}{E_c} = \frac{29000}{4031} = 7.2$$ (Eq. 1.6 in textbook)

From Textbook Section 3.3.b and Figure 3.5

$$b\frac{(kd)^2}{2} - nA_s(d - kd) = 0$$ (Eq. 3.5 in textbook)

substituting values and solving for *kd*:

$$12\frac{(kd)^2}{2} - (7.2)(3.81)(19.5 - kd) = 0$$

$$\rightarrow kd = 7.4 \text{ in}$$

$$jd = d - \frac{kd}{3} = 19.5 - \frac{7.4}{3} = 17 \text{ in} \qquad \text{(Fig. 3.5 in textbook)}$$

steel stress under service loading:

$$f_s = \frac{M}{A_s jd} = \frac{2104}{(3.81)(17)} = 32.5 \text{ ksi}$$

compare with the approximate ACI value (10.6.4): $f_s = 0.6 f_y = 36$ ksi (close)

the crack width under service loading (textbook section 6.1.b):

$$w = 0.076 \beta f_s \sqrt[3]{d_c A} \qquad \text{(Eq. 6.1 in textbook)}$$

$$\beta = \frac{h_2}{h_1} = \frac{h - kd}{d - kd} = \frac{22 - 7.4}{19.5 - 7.4} = 1.21 \qquad \text{(Fig. 6.1 in textbook)}$$

thus $w = (0.076)(1.21)(32.5)\sqrt[3]{2.5\frac{(5)(12)}{3}} = 11 \times 10^{-3}$ in

maximum crack width = 0.016 in > w OK

Check ACI provisions (Textbook Section 6.3)

Maximum reinforcement center-to-center spacing,

$$s = \frac{540}{f_s} - 2.5c_c \le 12\left(\frac{36}{f_s}\right)$$

$$= \frac{540}{32.5} - (2.5)(2.5 - \frac{1.27}{2}) \le 12\left(\frac{36}{32.5}\right)$$

$$= 12 \text{ in} \le 13.3 \text{ in} \qquad \rightarrow \quad s \le 12 \text{ in (obviously satisfied)}$$

As the variables affecting the crack width are (Textbook sections 6.2.a and b):

- Steel stress under service loading, f_s
- Concrete cover distance, d_c
- Ratio of distances from tension face and from steel centroid to neutral axis, β
- Concrete area surrounding one bar, A

Thus, decreasing the value of each of these parameters will decrease the crack width.

Problem 2 (Problem 3.11 in textbook)

$b = 24$ in

$h = 14$ in

$d = 11.5$ in

$f_c^{'} = 4{,}000$ psi

$f_y = 60{,}000$ psi

$A_s = 2\#11$ & $3\#10$

$A_s^{'} = 2\#10$

$d^{'} = 2.5$

Calculate the nominal and design strengths of the beam

(a) **ignore the compression steel**

reinforcement areas: $A_s = 2(1.56) + 3(1.27) = 6.93$ in^2 , $A_s^{'} = 2(1.27) = 2.54$ in^2

reinforcement ratio: $r = \frac{A_s}{bd} = \frac{6.93}{(24)(11.5)} = 0.0251$

balanced reinforcement ratio:

$$r_b = 0.85 b_1 \frac{f_c^{'}}{f_y} \frac{e_u}{e_u + e_y} = (0.85)^2 \frac{4}{60} \frac{0.003}{0.003 + \frac{60}{29000}} = 0.0285$$

minimum reinforcement ratio:

$$r_{\min} = \frac{3\sqrt{f_c^{'}}}{f_y} \; 3 \; \frac{200}{f_y}$$

$$= \frac{3\sqrt{4000}}{60000} \; 3 \; \frac{200}{60000}$$

$$= 0.0032 \; 3 \; 0.0033$$

$$r_{\min} = 0.0033$$

maximum reinforcement ratio:

$$r_{\max} = 0.75 r_b = 0.75(0.0285) = 0.0214$$

or $$r_{\max} = 0.85 b_1 \frac{f_c^{'}}{f_y} \frac{e_u}{e_u + 0.004} = (0.85)^2 \frac{4}{60} \frac{0.003}{0.007} = 0.0206$$

$\boxed{r > r_{\max}}$ The beam is overreinforced

Use the procedure in Textbook Section 3.4.g

$$m = \frac{E_s \varepsilon_u}{0.85 \beta_1 f_c'} = \frac{(29000)(0.003)}{(0.85)^2(4)} = 30.1 \qquad \text{(Eq. 3.43 in textbook)}$$

$$k_u^2 + m\rho k_u - m\rho = 0$$

$$k_u^2 + (30.1)(0.0251)k_u - (30.1)(0.0251) = 0$$

$$\Rightarrow \; k_u = 0.57$$

$$c = k_u d = (0.57)(11.5) = 6.56 \text{ in}$$

tension steel strain:

$$\varepsilon_s = \varepsilon_u \frac{d - c}{c} = 0.003 \frac{11.5 - 6.56}{6.56} = 0.00226$$

tension steel stress:

$$f_s = E_s \varepsilon_s = (29000)(0.00226) = 65.5 \text{ ksi}$$

Uh oh! $f_s > f_y$, which means $\varepsilon_s > \varepsilon_y$! But of course, after all, $\rho < \rho_b$.

HMM, what to do, what to do?

There are three alternative approaches:

1. Take steel yielding into account in the analysis, perform calculations for an underreinforced beam, but use ϕ=0.65 since it is overreinforced.
2. Release the condition that $\varepsilon_s \le \varepsilon_y$ and $f_s \le f_y$, use the procedure given in Section 3.4.g and assume that the safety is ensured by the low strength reduction factor ϕ=0.65.
3. Use the net tensile strength concept to calculate the proper ϕ factor, and use this factor in (1).

Approach 1 gets too conservative for reinforcement ratios closer to ρ_{max} than to ρ_b. Approach 2 gets nonconservative for high reinforcement ratios, close to ρ_b. Besides it violates the fundamental material property for steel, i.e. $f_s = f_y$ for $\varepsilon_s \ge \varepsilon_y$. Approach 3 is the best approach since a proper strength reduction factor ϕ is used after a proper reinforced concrete beam analysis.

Use the underreinforced beam formulation to calculate the nominal moment capacity since $\rho < \rho_b$.

$$M_n = \rho f_y b d^2 \left(1 - 0.59 \frac{\rho f_y}{f_c'}\right)$$

$$= (0.0251)(60)(24)(11.5^2)\left(1 - 0.59 \frac{(0.0251)(60)}{4}\right)$$

$$= 3718 \text{ kips-in}$$

To calculate the net tensile strain, e_t, use either

$$r = 0.85 b_1 \frac{f_c'}{f_y} \frac{e_u}{e_u + e_t}$$

$$0.0251 = 0.85^2 \frac{4}{60} \frac{0.003}{0.003 + e_t} \text{ Þ } e_t = 0.0028$$

or alternatively

$$c = \frac{A_s f_y}{0.85 b_1 f_c' b}, \qquad e_t = e_u \frac{d - c}{c}$$

$$c = \frac{(6.93)(60)}{0.85^2(4)(24)} = 6 \text{ in}, \qquad e_t = 0.003 \frac{11.5 - 6}{6} = 0.0028$$

From Fig. 3.9 in the textbook:

$$\begin{aligned} f &= 0.483 + 83.3 e_t \\ &= 0.483 + 83.3(0.0028) \\ &= 0.72 \end{aligned}$$

Finally calculate the design strength

$$M_u = f M_n = 0.72(3718) = 2677 \text{ kips-in}$$

(b) <u>Assume compression steel has yielded</u>

$$f_s' = f_y$$

$$r_{\max} = 0.0206 \quad \text{(calculated in a)}$$

$$r' = \frac{A_s'}{bd} = \frac{2.54}{(24)(11.5)} = 0.0092$$

$$\bar{r}_{\max} = r_{\max} + r' = 0.0206 + 0.0092 = 0.0298$$

$$\boxed{r = 0.0251 < \bar{r}_{\max}} \quad \text{The beam is underreinforced}$$

Calculate the nominal strength

$$a = \frac{(r - r') f_y d}{0.85 f_c'} = \frac{(0.0251 - 0.0092)(60)(11.5)}{0.85(4)} = 3.23 \text{ in}$$

$$c = \frac{a}{b_1} = \frac{3.23}{0.85} = 3.8 \text{ in}$$

$$M_n = M_{n1} + M_{n2} = A_s' f_y (d - d') + (A_s - A_s') f_y (d - \frac{a}{2})$$
$$= (2.54)(60)(11.5 - 2.5) + (6.93 - 2.54)(60)(11.5 - \frac{3.23}{2})$$
$$= 3975 \text{ kips-in}$$

Calculate the net tensile strength to determine ϕ

$$\varepsilon_t = \varepsilon_u \frac{d_t - c}{c} = 0.003 \frac{11.5 - 3.8}{3.8} = 0.0061 > 0.005 \Rightarrow \phi = 0.9$$

The design strength:

$$M_u = \phi M_n = 0.9(3975) = 3578 \text{ kips-in}$$

(c) <u>Compression steel not yielded</u>

Verify the compression steel does not yield

$$\bar{\rho}_{cy} = 0.85\beta_1 \frac{f_c'}{f_y} \frac{d'}{d} \frac{\varepsilon_u}{\varepsilon_u - \varepsilon_y} + \rho'$$
$$= 0.85^2 \frac{4}{60} \frac{2.5}{11.5} \frac{0.003}{0.003 - 0.0021} + 0.0092$$
$$= 0.044$$

$$\boxed{\rho = 0.0251 < \bar{\rho}_{cy}}$$ Compression steel does not yield

Check $\bar{\rho}_{max}$,

$$f_s' = E_s \left[\varepsilon_u - \frac{d'}{d}(\varepsilon_u + 0.004) \right] \le f_y$$
$$= 29000 \left[0.003 \frac{2.5}{11.5} (0.003 + 0.004) \right]$$
$$= 42.9 \text{ ksi} < f_y$$

$$\bar{\rho}_{max} = \rho_{max} + \rho' \frac{f_s'}{f_y} = 0.0206 + 0.0092 \frac{42.9}{60} = 0.0272$$

$\rho = 0.0251 < \bar{\rho}_{max}$ underreinforced beam

From force equilibrium

$$C + T' = T$$

$$0.85\beta_1 f_c' bc + A_s' E_s \varepsilon_s' = A_s f_y \ , \ \varepsilon_s' = \varepsilon_u \frac{c - d'}{c}$$

$$(0.85^2)(4)(24)c + (2.54)(29000)(0.003 \frac{c - 2.5}{c}) = (6.93)(60)$$

$$\Rightarrow c = 4.56 \text{ in} \ , \ a = \beta_1 c = 0.85(4.56) = 3.88 \text{ in}$$

Compression steel stress

$$f_s' = E_s \varepsilon_u \frac{c - d'}{c} = 29000(0.003)\frac{4.56 - 2.5}{4.56} = 39.3 \text{ ksi}$$

Nominal moment capacity

$$M_n = 0.85 f_c' ab(d - \frac{a}{2}) + A_s' f_s' (d - d')$$
$$= 0.85(4)(3.88)(24)(11.5 - \frac{3.88}{2}) + (2.54)(39.3)(11.5 - 2.5)$$
$$= 3925 \text{ kips-in}$$

Net tensile strain

$$\varepsilon_t = \varepsilon_u \frac{d_t - c}{c} = 0.003\frac{11.5 - 4.56}{4.56} = 0.0046 < 0.005 \Rightarrow \phi < 0.9$$

The strength reduction factor from Fig. 3.9 in the textbook:

$$\phi = 0.483 + 83.3\varepsilon_t$$
$$= 0.483 + 83.3(0.0046)$$
$$= 0.87$$

The design strength:

$$M_u = \phi M_n = 0.87(3925) = 3415 \text{ kips-in}$$

Problem 3 (Problem 3.12 in textbook)

$L = 20$ ft
$M_u = 5780$ kips-in
$b = 20$ in
$b_w = 10$ in
$h_f = 5$ in
$d = 20$ in
$f_y = 60$ ksi
$f_c' = 4$ ksi

Assume $a \le h_f$

From force equilibrium

$$C = T$$
$$0.85 f_c' b a = A_s f_y$$
$$0.85(4)(20)a = A_s(60)$$
$$\Rightarrow A_s = 1.133a$$

Moment capacity

$$M_n = A_s f_y \left(d - \frac{a}{2}\right)$$
$$\frac{M_u}{\phi} = (1.133a) f_y \left(d - \frac{a}{2}\right)$$
$$\frac{5780}{0.9} = (1.133a)(60)\left(20 - \frac{a}{2}\right)$$
$$33.99a^2 - 1359.6a + 6422.2 = 0$$
$$\Rightarrow a = 5.47 \text{ in}$$

$\boxed{a > h_f}$ Assumption does not hold

$a > h_f \Rightarrow a = h_f + t$

$$C = T$$
$$0.85 f_c' (b h_f + b_w t) = A_s f_y$$
$$0.85(4)[(20)(5) + 10t] = A_s(60)$$
$$\Rightarrow A_s = 0.567t + 5.67$$

To calculate the moment capacity, take moments of the concrete compression forces on the flange and the web around the axis of steel. (Note that this is different than the approach taken in the textbook, which divides the compression area into overhanging blocks and the rectangular stress block.)

$$M_n = 0.85 f_c^{'} b h_f (d - \frac{h_f}{2}) + 0.85 f_c^{'} b_w t (d - h_f - \frac{t}{2})$$

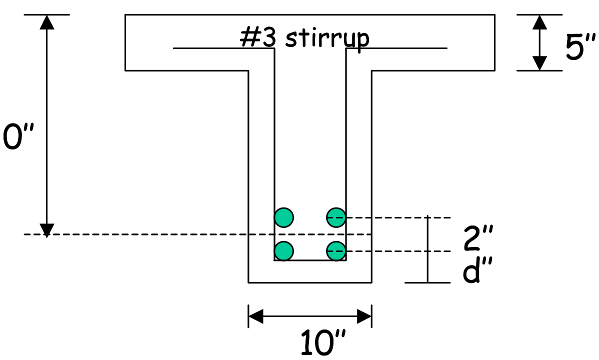
$$\frac{5780}{0.9} = 0.85(4)(20)(5)(20 - \frac{5}{2}) + 0.85(4)(10)t(20 - 5 - \frac{t}{2})$$

$$\Rightarrow t = 0.96 \text{ in}$$

$$\begin{aligned} A_s &= 0.567t + 5.67 \\ &= 0.567(0.96) + 5.67 \\ &= 6.21 \text{ in}^2 \end{aligned}$$

Use 4#11, $A_s = 4(1.56) = 6.24 \text{ in}^2$

According to the Appendix Table A.7 in the textbook, only 2#11 can be placed in a single row for $b = 10$ in . Thus, need to use two rows of reinforcement



$d" =$ half bar diameter + stirrup diameter + clear cover

$$= \frac{1.41}{2} + \frac{3}{8} + 1.5$$

$$= 2.6 \text{ in} \approx 3 \text{ in}$$

Thus, $h = 20 + 1 + 3 = 24$ in

WHEW!!!