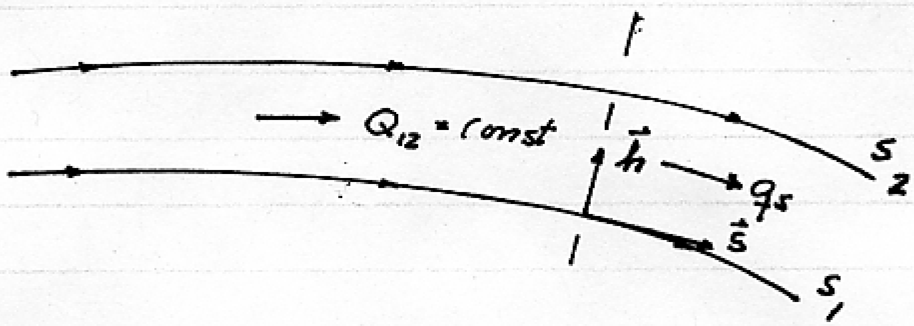
LECTURE #6

1.060 ENGINEERING MECHANICS II

Conservation of Volume for a Streamtube (2-D Plane Flow)



Streamline coordinates:

$\vec{s}$ is local direction of streamline
$\vec{h}$ is local direction ⊥ streamline

and

$$\vec{q}(s) = (q_s, q_h) = (q_s, 0) = q_s$$

STREAM FUNCTION

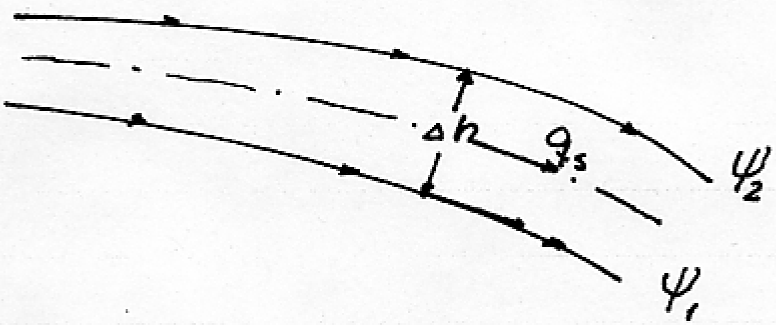
Define a function ψ such that

$$\frac{\partial \psi}{\partial s} = 0 \quad , \quad \frac{\partial \psi}{\partial h} = q_s$$

and therefore

$$\psi = \psi(s,h) = \psi(h) = \text{constant along } \vec{s}$$

ψ = stream function is constant along a stream line.



$$\frac{\partial \psi}{\partial h} = \lim_{\Delta h \to 0} \frac{\Delta \psi}{\Delta h} \approx \frac{\psi_2 - \psi_1}{\Delta h} \approx q_s$$

$$\underline{\psi_2 - \psi_1 = q_s \, \Delta h = Q_{12} = \text{discharge in tube 1-2}}$$

VELOCITY POTENTIAL

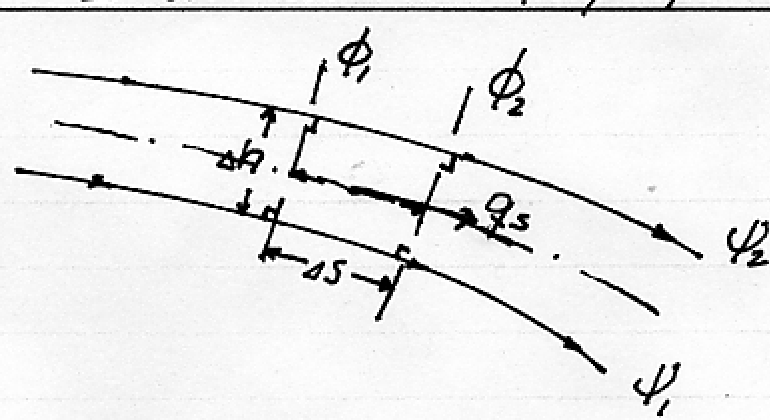
Define a function ϕ such that

$$\frac{\partial \phi}{\partial s} = q_s \quad , \quad \frac{\partial \phi}{\partial h} = 0$$

and therefore

$$\phi = \phi(s, h) = \phi(s) = \text{constant along lines} \perp \vec{s}$$

<u>ϕ = velocity potential is constant along lines $\perp$ streamlines (equipotential lines)</u>



$$\frac{\partial \phi}{\partial s} = \lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s} \approx \frac{\phi_2 - \phi_1}{\Delta s} \cong q_s$$

or

$$\phi_2 - \phi_1 = q_s \Delta s = (q_s \Delta h) \frac{\Delta s}{\Delta h} = Q_{12}$$

if $\Delta s / \Delta h \approx 1$

FLOW NET

Lines of constant ψ (streamlines) and constant ϕ (equipotential lines) form two families of curves (s-lines and h-lines) that intersect each other at right angles, i.e. they form a "rectangular" pattern.

If the two families of curves are constructed (drawn) such that they form a pattern of "squares", i.e. $\Delta s = \Delta h$, the discharge in stream tubes formed by adjacent stream lines, Q_{12}, is constant (as it should be). This is a FLOW NET

If we know the discharge, Q_{12}, in a stream tube, we can obtain an estimate of the velocity at the center of a "square" from $q_s \cong Q_{12} / \Delta h$, where Δh ($= \Delta s$) is the side length of the square.

Simple "Rules" to follow when
<u>Constructing Flow Nets</u>

1) <u>Use a pencil and bring an eraser.</u>

2) Solid boundaries are streamlines

3) Region(s) where flow is uniform - straight parallel streamline - is a good place to start and end.

4) Sketch the streamlines and equipotential lines connecting region(s) identified in (3) observing the following
   Ψ-& Φ-lines form a "square" pattern
   Streamlines can never cross
   Streamlines can start & end only at flow areas
   Streamlines in the interior are guided by boundaries, but smoother (no kinks)

5) Examine sketch and "repair" where needed (this is where the eraser comes in!)

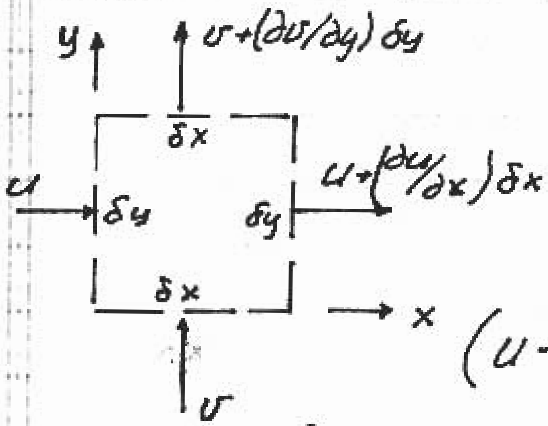
6) Discharge $= \Delta Q$ is constant within a tube
$V = \Delta Q / \Delta h$ can be estimated at center of "squares". Directed along Ψ.

7) Pressure is estimated from Bernoulli along a streamline.

STREAM FUNCTION & VELOCITY POTENTIAL

Their Generalized Definitions and Limitations

Differential Form of Continuity Equation



Volume Conservation:
Rate in - Rate out = Rate of change = 0 if incompressible fluid.

$$\left(u-\left(u+\left(\partial u/\partial x\right)\delta x\right)\right)\delta y+\left(v-\left(v+\left(\frac{\partial v}{\partial y}\right)\delta y\right)\right)\delta x=0$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \text{in 2-D}$$

or

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla\cdot\vec{q}=0 \quad \text{in 3-D}$$

Stream Function, ψ

$$\psi=\psi(x,y,t)$$

defined by

$$u=\partial\psi/\partial y \; ; \; v=-\partial\psi/\partial x$$

Continuity equation (in 2-D) is automatically satisfied, since

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right)=\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial^2\psi}{\partial y\partial x}\equiv 0$$

Limitation of Stream Function

Two-Dimensional Flows of an Incompressible Fluid

Velocity Potential, ϕ

$$\phi = \phi(x, y, z, t)$$

defined by

$$u = \partial\phi/\partial x \; ; \; v = \partial\phi/\partial y \; ; \; w = \partial\phi/\partial z$$

or

$$\vec{q} = (u, v, w) = \text{grad}\,\phi = \nabla\phi$$

In terms of ϕ, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) =$$

$$\underline{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi = 0} \quad \text{(Laplace Eq.)}$$

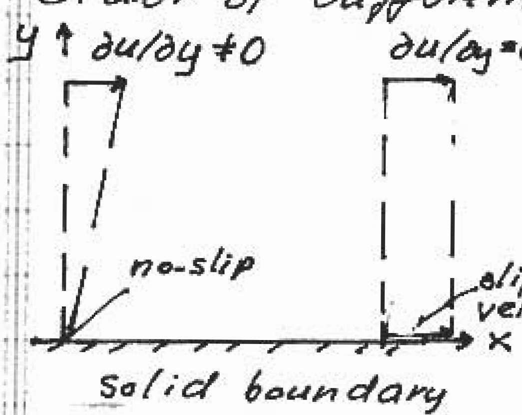
$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} = \text{Laplace Operator}$$

Limitation of Velocity Potential

For ϕ to make sense, we must have that

$$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial v}{\partial x} = \frac{\partial^2\phi}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial u}{\partial y}$$

Order of differentiation is immaterial!



At solid boundary $v = 0$,

i.e. $\partial v/\partial x = 0$

So, for ϕ to exist,

$\partial u/\partial y = 0$

which cannot be the case unless fluid is inviscid ($\nu = 0$)

FLOW NET LIMITATIONS

Since flow nets are based on Stream Function and Velocity Potential concepts they are limited to:

<u>Two-Dimensional</u> stedd<u>y Flows</u> of an <u>Incompressible</u> and <u>Inviscid</u> Fluid,

or

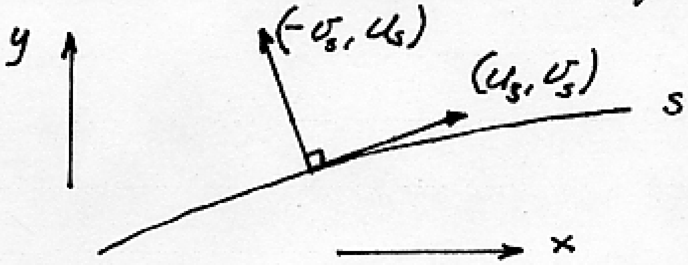
<u>Two-Dimensional steady Flows of an Ideal Fluid</u>

- We can still sketch flow nets for 3-D flows and get a very good physical picture of the nature of the flow, e.g. regions where velocities are large or small may be identified.

- The flow net allows fluid to have a slip-velocity along solid boundaries (they become streamlines!). Since there is no such thing as an ideal fluid the <u>flow net features very close to solid boundaries are unreliable</u>.

- Flow nets are useful when the bulk of the flow may be considered minimally affected by boundary shear stresses, and the flow is "converging", i.e. velocity is increasing along a streamline.

Stream Line Coordinates from Cartesian



$$(u_s, v_s) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$$

$$(u_s, v_s) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)$$

$$\frac{\partial \psi}{\partial s}\delta s = \frac{\partial \psi}{\partial x}\delta x + \frac{\partial \psi}{\partial y}\delta y = \frac{\partial \psi}{\partial x}(u_s \delta t) + \frac{\partial \psi}{\partial y}(v_s \delta t) =$$

$$\left[\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)\right]\delta t \equiv 0$$

i.e.

$$\underline{\partial \psi / \partial s = 0 \quad \text{along streamlines}}$$

In the direction $\perp s$, i.e. the "h"-direction

$$\frac{\partial \psi}{\partial h}\delta h = \frac{\partial \psi}{\partial h}(u_s^2 + v_s^2)^{1/2}\delta t = \frac{\partial \psi}{\partial x}(-v_s \delta t) + \frac{\partial \psi}{\partial y}(u_s \delta t) =$$

$$\left[\frac{\partial \psi}{\partial x}\left(-\left(-\frac{\partial \psi}{\partial x}\right)\right) + \frac{\partial \psi}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)\right]\delta t = (v_s^2 + u_s^2)\delta t$$

or

$$\underline{\partial \psi / \partial h} = (u_s^2 + v_s^2)^{1/2} = \underline{q_s} = \text{vel. in s-direct.}$$

Similarly,

$$\frac{\partial \phi}{\partial s}\delta s = \frac{\partial \phi}{\partial s}(u_s^2 + v_s^2)^{1/2}\delta t = \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y = (u_s^2 + v_s^2)\delta t$$

$$\underline{\partial \phi / \partial s} = (u_s^2 + v_s^2)^{1/2} = \underline{q_s}$$

and

$$\frac{\partial \phi}{\partial h}\delta h = \frac{\partial \phi}{\partial x}(-v_s \delta t) + \frac{\partial \phi}{\partial y}(u_s \delta t) = (-u_s v_s + u_s v_s)\delta t = 0$$

$$\underline{\partial \phi / \partial h = 0}$$