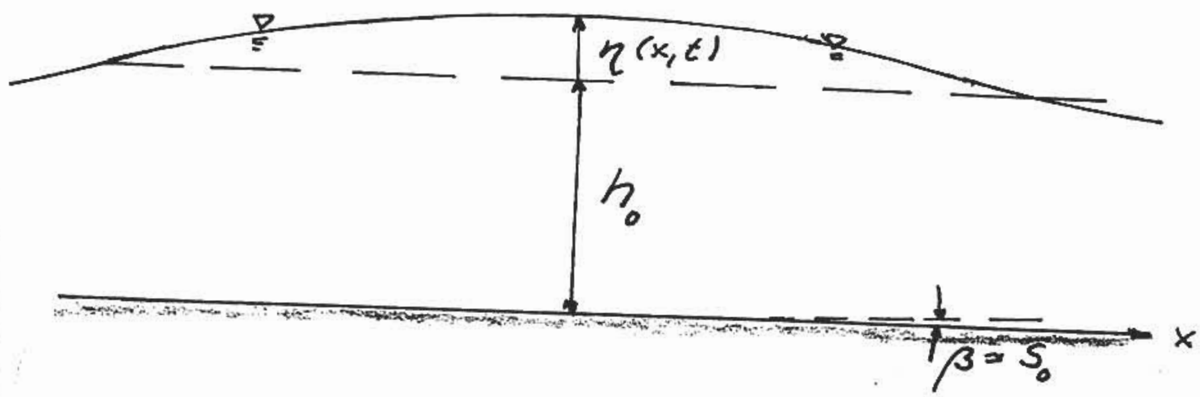
RECITATION #8

1.060 ENGINEERING MECHANICS II

UNSTEADY FLOW IN OPEN CHANNELS



We consider a free surface flow in a prismatic channel of small, but constant, slope, i.e. with h denoting the depth

$$A = A(h) \; ; \; P = P(h) \; ; \; R_h = \frac{A}{P} = R_h(h) \qquad (1)$$

and

$$\sin\beta \approx \beta = S_o = \text{constant} \; ; \; \cos\beta \approx 1.0 \qquad (2)$$

The depth of flow, h, is expressed in terms of its deviation from normal, i.e. steady uniform flow, for an assumed base flow discharge Q_o

$$h = h_o + \eta(x,t) \qquad (3)$$

where h_o = constant

Continuity Equation

Volume conservation for an unsteady flow requires

$$Q_{in} - Q_{out} = \frac{\partial}{\partial t}(\text{Volume between "in" and "out"})$$

Taking our CV as a slice of the channel of infinitesimal length δx, this becomes

$$Q - \left(Q + \frac{\partial Q}{\partial x}\delta x\right) = \frac{\partial}{\partial t}(A\delta x)$$

or

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{4}$$

For a prismatic channel we have, using (3),

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial h}\frac{\partial h}{\partial t} = b_s \frac{\partial \eta}{\partial t} \tag{5}$$

and

b_s = surface width of channel = $b_s(h)$

With

$$Q = VA$$

we also have that

$$\frac{\partial Q}{\partial x} = \frac{\partial (VA)}{\partial x} = A\frac{\partial V}{\partial x} + V\frac{\partial A}{\partial x} = A\frac{\partial V}{\partial x} + V\frac{\partial A}{\partial h}\frac{\partial h}{\partial x} =$$

$$A\frac{\partial V}{\partial x} + V b_s \frac{\partial \eta}{\partial x} \tag{6}$$

Introducing (5) & (6) in (4) yields an alternative form of the continuity equation

$$\frac{\partial \eta}{\partial t} + V \frac{\partial \eta}{\partial x} + h_m \frac{\partial V}{\partial x} = 0 \qquad (7)$$

where

$$h_m = \text{mean depth} = \frac{A}{b_s} = h_m(h) \qquad (8)$$

Momentum Equation

The momentum principle in the direction parallel to be bottom (the x-direction) gives

$$MP_{in} - MP_{out} + (\text{Boundary Shear Forces})_x + (\text{Gravity})_x =$$
$$\frac{\partial}{\partial t}(\text{x-momentum in volume between "in" and "out"})$$

Again, choosing a slice of length δx, gives

$$MP - \left(MP + \frac{\partial MP}{\partial x}\delta x\right) - \tau_s P \delta x + \rho g A \delta x \sin\beta =$$
$$\frac{\partial}{\partial t}(\rho \underbrace{VA}_{Q} \delta x)$$

or

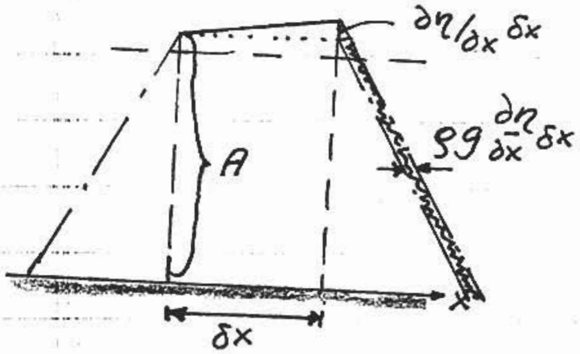
$$\rho \frac{\partial Q}{\partial t} + \frac{\partial MP}{\partial x} + \tau_s P - \rho g A S_0 = 0 \qquad (9)$$

We have that

$$MP = (\rho V^2 + p_{cG})A = \rho Q V + P_x$$

where P_x = total pressure force on A in x-direction, and therefore

$$\frac{\partial MP}{\partial x} = \rho \frac{\partial (QV)}{\partial x} + \frac{\partial P_x}{\partial x}$$



The evaluation of $\frac{\partial P_x}{\partial x}$ is facilitated by examination of pressure distribution on the "in" and "out" areas of our CV, and is found to be

$$\frac{\partial P_x}{\partial x} = \rho g A \frac{\partial \eta}{\partial x}$$

Thus, the contribution of the thrust terms (9) becomes

$$\frac{\partial MP}{\partial x} = \rho \left(\frac{\partial (QV)}{\partial x} + g A \frac{\partial \eta}{\partial x} \right) \qquad (10)$$

For the boundary shear stress we choose the expression consistent with the use of Manning's Equation, i.e. from Lecture #24 we have

$$\tau_s = \rho g n^2 V^2 / R_h^{1/3} \qquad (11)$$

where 'n' = Manning's 'n'.

Introducing (10) & (11) in (9) gives us

$$\rho \frac{\partial Q}{\partial t} + \rho \frac{\partial (QV)}{\partial x} + \rho g A \frac{\partial \eta}{\partial x} + \rho g n^2 V^2 \frac{P}{R_h^{1/3}} - \rho g A S_0 = 0$$

but

$$\frac{\partial Q}{\partial t} + \frac{\partial (QV)}{\partial x} = \frac{\partial (AV)}{\partial t} + \frac{\partial (QV)}{\partial x} = V\left(\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x}\right)$$

$$+ A\frac{\partial V}{\partial t} + Q\frac{\partial V}{\partial x} = A\frac{\partial V}{\partial t} + VA\frac{\partial V}{\partial x}$$

since $(\partial A/\partial t + \partial Q/\partial x) = 0$ by virtue of (4).

Thus, the momentum equation may be written as

$$\underbrace{\frac{\partial V}{\partial t}} + \underbrace{V\frac{\partial V}{\partial x} + g\frac{\partial \eta}{\partial x}} + \underbrace{g\frac{n^2}{R_h^{4/3}}V^2} - \underbrace{gS_0} = 0 \qquad (12)$$

unsteady, non-uniform, frictional, sloping-Flow.

Notice, that (12) corresponding to uniform steady flow, which is the base flow condition, reduces to Manning's Equation

$$V_0 = \frac{1}{n} R_{h0}^{2/3} S_0^{1/2} \qquad (13)$$

and $Q_0 = V_0 A_0$ is the base flow discharge.

The combination of Continuity, eq. (7), and Momentum, eq. (12), constitutes the two equations from which the two unknows, $V = Q/A$ and $h = h_0 + \eta$, may be determined in the most general case of unsteady, non-uniform flow in a sloping prismatic channel of known geometry and roughness.

Linearization Technique

The governing equations, (7) and (12), are hopelessly NON-LINEAR and defy simple analytical solution techniques. They may be solved numerically, but it is often desirable to obtain an approximate analytical solution, since such a solution often reveals the physical nature of the solution (and in addition may serve as a limiting case against which a numerical solution technique may be tested for accuracy and convergence).

To achieve this goal, we limit the deviations from the base flow to be small relative to the characteristic base flow quantities. In this case, we therefore take

$$\left.\begin{aligned} V &= V_0 + v(x,t) \\ h &= h_0 + \eta(x,t) \end{aligned}\right\} \quad (14)$$

and assume that

$$\frac{|v(x,t)|}{V_0} \approx \frac{\eta(x,t)}{h_0} \approx O(\varepsilon) \ll 1 \qquad (15)$$

so that terms of order ε^2 may be assumed negligibly small.

In this manner all NON-LINEAR terms, involving products of η and υ, are removed from the governing equations and these became linear in η and υ.

For the parameters defined by the prismatic channel's geometry, we have

$$A(h) = A_0 + \left.\frac{\partial A}{\partial h}\right|_{h_0}(h-h_0)+\dots \cong A_0 + b_{s0}\,\eta = A_0\left(1+\frac{\eta}{h_{m0}}\right)$$

$$b_s(h) = b_{s0} + \left.\frac{\partial b_s}{\partial h}\right|_{h_0}(h-h_0)+\dots \cong b_{s0} + \alpha_b\,\eta = b_{s0}\left(1+\alpha_b\frac{\eta}{b_{s0}}\right)$$

Therefore

$$h_m(h) = \frac{A(h)}{b_s(h)} = \frac{A_0\left(1+\eta/h_{m0}\right)}{b_{s0}\left(1+\alpha_b\frac{\eta}{b_{s0}}\right)} \cong h_{m0}\left(1+\left(1-\alpha_b\frac{h_{m0}}{b_{s0}}\right)\frac{\eta}{h_{m0}}\right) \tag{16}$$

where we used that

$$(1+\varepsilon)^{-1} = 1-\varepsilon+O(\varepsilon^2)$$

Similarly, we have

$$P(h) = P_0 + \left.\frac{\partial P}{\partial h}\right|_{h_0}(h-h_0)+\dots \cong P_0 + \alpha_p\,\eta = P_0\left(1+\alpha_p\frac{\eta}{P_0}\right)$$

so that

$$R(h) = \frac{A(h)}{P(h)} \cong \frac{A_0\left(1+\eta/h_{m0}\right)}{P_0\left(1+\alpha_p\,\eta/P_0\right)} \cong R_{h0}\left(1+\left(1-\alpha_p\frac{h_{m0}}{P_0}\right)\frac{\eta}{h_{m0}}\right) \tag{17}$$

Solution of Linearized Equations for the Frictionless Case

Introducing (14) and (16) in (7) gives

$$\frac{\partial \eta}{\partial t} + (V_o + v)\frac{\partial \eta}{\partial x} + \Big[h_{mo} + \Big(1 - \alpha_b \frac{h_{mo}}{b_{so}}\Big)\underset{O(\varepsilon)}{\frac{\eta}{h_{mo}}}\Big]\underset{O(\varepsilon)}{\frac{\partial (V_o + v)}{\partial x}} =$$

$$\frac{\partial \eta}{\partial t} + V_o \frac{\partial \eta}{\partial x} + h_{mo}\frac{\partial v}{\partial x} = O(\varepsilon^2) \cong 0 \qquad (18)$$

For the frictionless case 'n' = 0 and $S_o = 0$, i.e. horizontal bottom, and (12) becomes

$$\frac{\partial (V_o + v)}{\partial t} + \underset{O(\varepsilon)}{(V_o + v)}\underset{O(\varepsilon)}{\frac{\partial (V_o + v)}{\partial x}} + g\frac{\partial \eta}{\partial x} =$$

$$\frac{\partial v}{\partial t} + V_o\frac{\partial v}{\partial x} + g\frac{\partial \eta}{\partial x} = O(\varepsilon^2) \cong 0 \qquad (19)$$

Equations (18) and (19) are the two linearized equations for the two unknowns, $\eta(x,t)$ and $v(x,t)$. But v and η are involved in <u>both</u> equations. This, however, is readily ~~un~~remedied by getting

$$h_{mo}\frac{\partial v}{\partial x} = -\Big(\frac{\partial \eta}{\partial t} + V_o\frac{\partial \eta}{\partial x}\Big)$$

from (18), and therefore

$$\left.\begin{aligned} h_{mo}\frac{\partial^2 v}{\partial x \partial t} &= -\Big(\frac{\partial^2 \eta}{\partial t^2} + V_o\frac{\partial^2 \eta}{\partial x \partial t}\Big) \\ h_{mo}\frac{\partial^2 v}{\partial x^2} &= -\Big(\frac{\partial^2 \eta}{\partial x \partial t} + V_o\frac{\partial^2 \eta}{\partial x^2}\Big) \end{aligned}\right\} \qquad (20)$$

Upon multiplication of (19) by h_{mo}, followed by differentiation with respect to x, we have

$$h_{mo}\frac{\partial^2 \bar{v}}{\partial x \partial t} + V_0 h_{mo}\frac{\partial^2 \bar{v}}{\partial x^2} + g h_{mo}\frac{\partial^2 \bar{\eta}}{\partial x^2} = 0$$

and introducing (20) eliminates 'v' from the equation that now is exclusively in $\eta(x,t)$ = the free surface deviation from the base flow location

$$-\left(\frac{\partial^2 \bar{\eta}}{\partial t^2} + V_0\frac{\partial^2 \bar{\eta}}{\partial x \partial t}\right) + V_0\left(-\frac{\partial^2 \bar{\eta}}{\partial x \partial t} - V_0\frac{\partial^2 \bar{\eta}}{\partial x^2}\right) + g h_{mo}\frac{\partial^2 \bar{\eta}}{\partial x^2} = 0$$

or

$$\frac{\partial^2 \bar{\eta}}{\partial t^2} + 2V_0\frac{\partial^2 \bar{\eta}}{\partial x \partial t} - \left(g h_{mo} - V_0^2\right)\frac{\partial^2 \bar{\eta}}{\partial x^2} = 0 \qquad (21)$$

To obtain a solution to this equation we assume a solution of the form

$$\eta(x,t) = \eta(x-ct) = \eta(\theta) \qquad (22)$$

which represents a surface that translates at a speed 'c' in the x-direction, which a change in shape, i.e. of permanent form. Note, if $c>0$ the "wave" moves in $+x$-direction whereas it moves in the minus $(-)$ x-direction if $c<0$. $\theta = x-ct$ is referred to as the phase function.

Now, from (22) it follows that

$$\left.\begin{aligned}
\frac{\partial \eta}{\partial x} &= \frac{d\eta}{d\theta}\frac{\partial \theta}{\partial x} = \frac{d\eta}{d\theta}\cdot 1 = \eta' \\
\frac{\partial^2 \eta}{\partial x^2} &= \frac{d^2\eta}{d\theta^2}\cdot\frac{\partial \theta}{\partial x} = \frac{d^2\eta}{d\theta^2}\cdot 1 = \eta'' \\
\frac{\partial^2 \eta}{\partial x \partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x}\right) = \frac{\partial}{\partial \theta}\left(\frac{d\eta}{d\theta}\right)\cdot\frac{\partial \theta}{\partial t} = \eta''(-c) = -c\eta'' \\
\frac{\partial^2 \eta}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial t}\right) = \frac{d}{d\theta}\left(\frac{d\eta}{d\theta}\frac{\partial \theta}{\partial t}\right)\frac{\partial \theta}{\partial t} = c^2\eta''
\end{aligned}\right\} \quad (23)$$

Introducing these expressions in (21) we obtain

$$c^2\eta'' + 2V_0(-c\eta'') - (gh_{mo} - V_0^2)\eta'' = 0 \qquad (24)$$

Note: No matter what $\eta'' = d^2\eta/d\theta^2$ is it cancels out since it is common to all terms. Thus, (24) becomes an equation for 'c' = the speed of the disturbance

$$c^2 - 2V_0 c - (gh_{mo} - V_0^2) = 0$$

or

$$c = V_0 \pm \sqrt{gh_{mo} - V_0^2 + V_0^2} = V_0 \pm \sqrt{gh_{mo}} \qquad (25)$$

Therefore our assumed form of our solution is indeed a solution, which can be written as

$$\eta = \eta_+\left(x + \left(\sqrt{g h_{mo}} - V_0\right)t\right) + \eta_-\left(x - \left(\sqrt{g h_{mo}} + V_0\right)t\right) \tag{26}$$

i.e. a wave traveling upstream, η_+, whenever

$$\sqrt{g h_{mo}} - V_0 > 0 \Rightarrow Fr_0 = \frac{V_0}{\sqrt{g h_{mo}}} < 1 \tag{27}$$

i.e. for subcritical flow.
For supercritical flow, $Fr_0 > 1$, both η_+ and η_- travel in the downstream direction.

From our general solution for η, (22), and (20), we obtain

$$\frac{\partial v}{\partial x} = \frac{dv}{d\theta}\frac{\partial \theta}{\partial x} = \frac{dv}{d\theta} = v' =$$

$$-\frac{1}{h_{mo}}\left(\frac{\partial \eta}{\partial t} + V_0 \frac{\partial \eta}{\partial x}\right) = -\frac{1}{h_{mo}}\left(-c + V_0\right)\eta' =$$

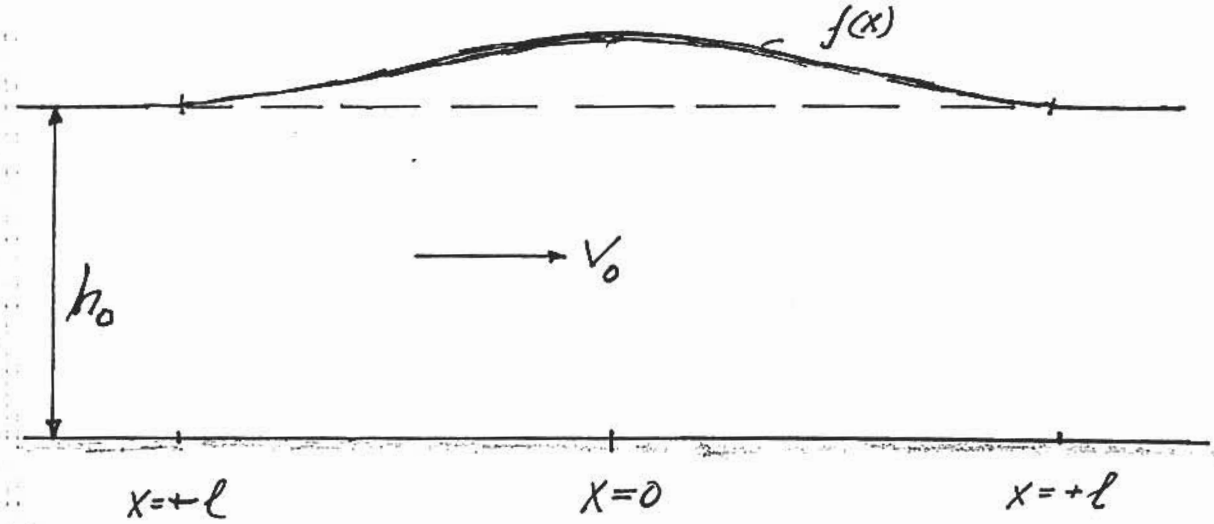
or, with c given by (25)

$$v' = \frac{c - V_0}{h_{mo}}\,\eta' \Rightarrow v(\theta) = \frac{\pm\sqrt{g h_{mo}}}{h_{mo}}\,\eta(\theta) \tag{28}$$

The solution for v corresponding to (26) therefore is

$$v = v_+ + v_- = -\frac{\eta_+}{h_{mo}}\sqrt{g h_{mo}} + \frac{\eta_-}{h_{mo}}\sqrt{g h_{mo}} \tag{29}$$

Special Solution to Special Problem



Imagine that we 'dump' a volume of water into our base flow at $t=0$, i.e.

$$t \leq 0 \quad : \quad \eta = 0 \ \& \ v = 0 \text{ for all } x$$

$$t = 0 \qquad \eta = \begin{cases} f(x) & |x| < \ell \\ 0 & |x| > \ell \end{cases} \quad \& \ v = 0 \text{ for all } x$$

From our general solution for η we have at $t=0$ from (26)

$$\eta = \eta_+(x) + \eta_-(x) = f(x) \qquad (t=0;\ |x|<\ell)$$

and, from (29),

$$v = v_+ + v_- = \left(-\eta_+(x) + \eta_-(x)\right)\frac{\sqrt{g h_{mo}}}{h_{mo}} = 0$$

Obviously

$$\eta_+(x) = \eta_-(x) = \frac{1}{2} f(x) \qquad (|x|<\ell)$$

does the job

Thus, the initial disturbance of the base flow will split into to waves that start out of identical shape at $t = 0$.

$$\eta_+(x) = \eta_-(x) = \begin{cases} f(x)/2 & |x| < l \\ 0 & |x| > l \end{cases}$$

One is moving in the upstream direction with a surface profile given by

$$\eta_+ = \frac{1}{2} f\left(x + (\sqrt{g h_{mo}} - V_0)t\right) \quad \left|x + (\sqrt{g h_{mo}} - V_0)t\right| < l$$

the other moving downstream

$$\eta_- = \frac{1}{2} f\left(x - (\sqrt{g h_{mo}} + V_0)t\right) \quad \left|x - (\sqrt{g h_{mo}} + V_0)\right| < l$$

provided

$$Fr_0 = \frac{V_0}{\sqrt{g h_{mo}}} < 1.$$

<u>IN SUBCRITICAL FLOWS DISTURBANCES PROPAGATE AND ARE AFFECTING CONDITIONS UPSTREAM OF LOCATION OF DISTURBANCE</u>

<u>IN SUPERCRITICAL FLOWS DISTURBANCES PROPAGATE AND AFFECT CONDITIONS DOWNSTREAM OF DISTURBANCE</u>