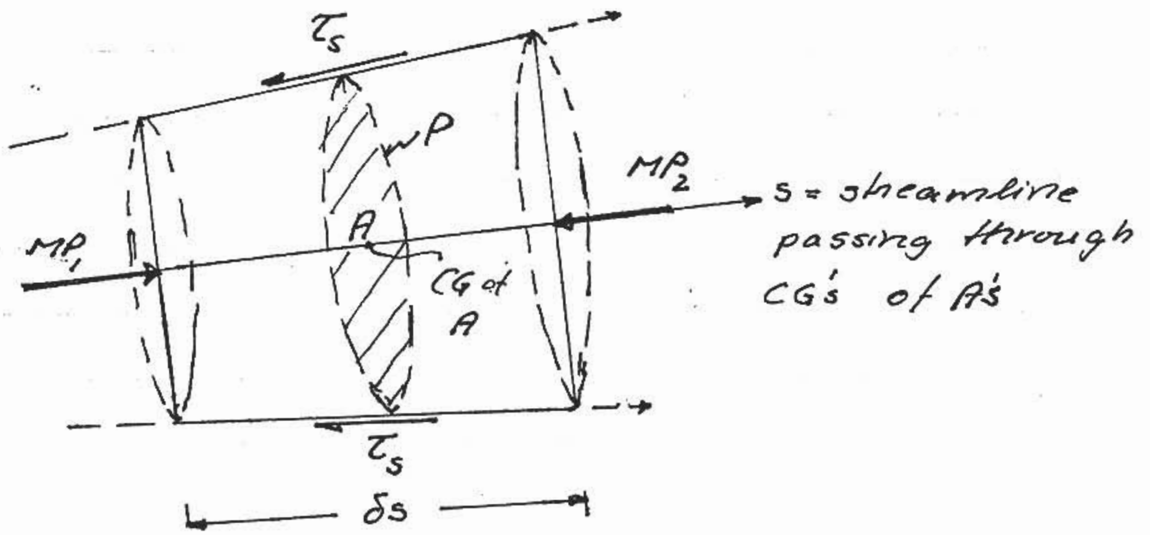
LECTURE #12

1.060 ENGINEERING MECHANICS II

MOMENTUM PRINCIPLE for a Streamtube



In s-direction we have (steady flow, ρ=const.)

$$(MP_1 - MP_2) + (\text{Gravity Component}) + (\text{Other Forces in } s) = 0$$

$$\underline{MP_1 - MP_2} = -\frac{\partial MP}{\partial s}\delta s = -\frac{\partial}{\partial s}(\rho \underset{V_s A = const.}{Q} V_s + p_{CG} A)\delta s =$$

$$-\left(\rho Q \frac{\partial V_s}{\partial s} + A \frac{\partial p_{CG}}{\partial s}\right)\delta s = \underline{-\frac{\partial}{\partial s}\left(\frac{1}{2}\rho V_s^2 + p_{CG}\right) A \delta s}$$

$$\underline{\text{Gravity}} = (\rho A \delta s)\, g_s = -\rho g \frac{\partial z_s}{\partial s} A \delta s = \underline{-\frac{\partial}{\partial s}(\rho g z_s) A \delta s}$$

<u>Other forces</u> = Shear stresses acting on perimeter of streamtube, P, multiplied by area = $\underline{-\tau_s P \delta s}$ (τ_s = AVERAGE SHEAR STRESS ON STREAMTUBE WALLS TAKEN POSITIVE IF ACTING IN DIRECTION OPPOSITE OF s, I.E. OPPOSITE TO V_s)

Thus, the momentum principle becomes

$$-\frac{\partial}{\partial s}\left(\frac{1}{2}\rho V_s^2 + p_{CG} + \rho g z_{CG}\right)A\delta s - \tau_s P\,\delta s = 0$$

or

$$\frac{\partial}{\partial s}\left(\frac{1}{2}\rho V_s^2 + p_{CG} + \rho g z_{CG}\right) = -\tau_s \frac{P}{A}$$

when integrated along s from s_1 to s_2

$$\underline{\left[\frac{1}{2}\rho V_s^2 + p_{CG} + \rho g z_{CG}\right]_{s_1}^{s_2} = -\int_{s_1}^{s_2} \tau_s \frac{P}{A}\,ds}$$

In terms of the Bernoulli Head, H, this is

$$\underline{H_2 - H_1 = -\int_{s_1}^{s_2} \tau_s \frac{P}{A\rho g}\,ds} \quad \text{or} \quad \underline{\frac{\partial H}{\partial s} = -\frac{\tau_s P}{\rho g A}}$$

where

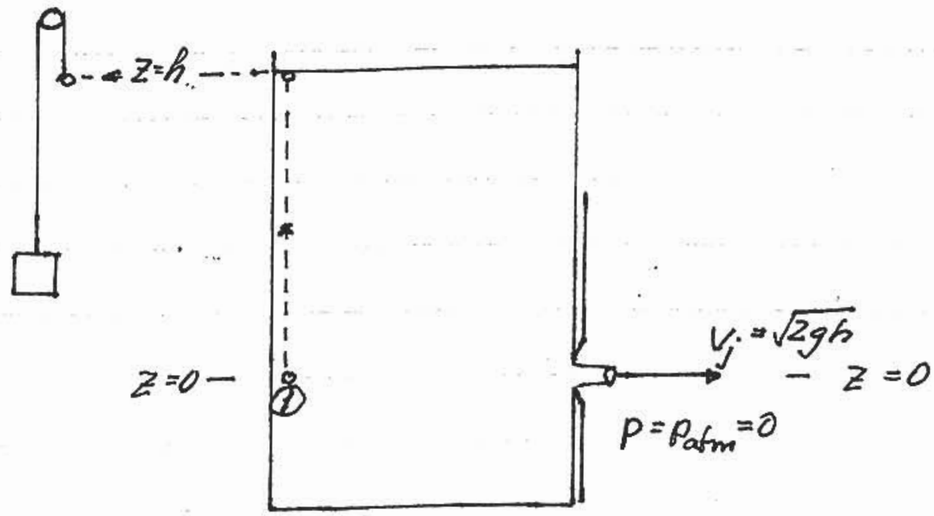
$$\underline{H = \frac{V^2}{2g} + \frac{p_{CG}}{\rho g} + z_{CG}} \quad \& \quad \underline{V = \frac{Q}{A}}$$

Note: If $\tau_s = 0$, i.e. for a frictionless flow, $H_2 = H_1$ and we have the Bernoulli Equation along a streamline.

Note also: Momentum Coefficient, $K_{m_1} = \int_A q_L^2\,dA/(AV^2)$, was assumed to be unity (or could be incluedea as a constant). This means that the Bernoulli Equation above is acceptable only if <u>flow is well behaved everywhere</u> along the streamtube leading from s_1 to s_2.

BERNOULLI EQUATION FROM ENERGY

If you have ENERGY, you can do WORK



Far away from orifice $p + \rho g z = \text{constant} = \rho g h$ since $\vec{q} \approx 0$.

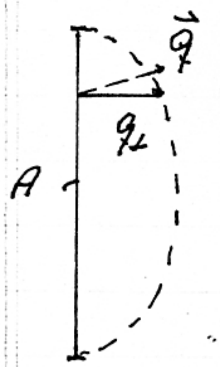
Since a small volume of fluid in the bucket is neutrally buoyant, one can move the particle denoted by 1 from $z=0$ up to $z=h$ without doing any work against gravity. Once at $z=h$, the particle can be moved horizontally ($\perp \vec{g}$ and hence no work required) outside the bucket where it now can be used to do work (e.g. by hoisting a weight through a pulley system) We can do this for any particle in the bucket, i.e. the fluid in the bucket possesses a "potential" potential energy $\rho g h = p + \rho g z$ per unit volume.

A fluid particle leaving the bucket through the orifice has no "potential" potential energy as it passes vena contracta ($p = p_{atm} = 0$ and $z = 0$). It does, however, have kinetic energy since it is moving at a velocity V_j. From Bernoulli we have $V_j = \sqrt{2gh}$, and the kinetic energy per unit volume of fluid leaving the bucket through vena contracta is therefore $\frac{1}{2}\rho V_j^2 = \rho g h$.

Considering the bucket up to the vena contracta as a "system" it is clear that the system loses potential energy, $\rho g h \, \delta V$, when a volume δV exits through vena contracta, whereas the "outside world" gains the kinetic energy of the exiting volume, $\frac{1}{2}\rho V_j^2 \, \delta V = \rho g h \, \delta V$ = loss of system's potential energy.

From the preceding discussion it follows that the Bernoulli Equation can be considered to express that the Mechanical Energy of a fluid particle remains constant as it moves about (without friction!)

$$\underbrace{\tfrac{1}{2}\rho \bar{q}^{2}}_{\text{Kinetic Energy}} + \underbrace{p + \rho g z}_{\text{"Potential" potential Energy}} = \text{Mech. Energy per unit volume of fluid} = m_e$$



$\dot{E}$ = rate of mechanical energy flow across area A =

$$\int_A \left(\tfrac{1}{2} \rho \vec{q}^{\,2} + p + \rho g z \right) q_\perp \, dA$$

If flow is well behaved and $A \perp$ straight streamlines, then:

$$\vec{q} \approx q_\perp \quad \text{and} \quad p + \rho g z = p_{CG} + \rho g z_{GG} = \text{const.}$$

and

$$\dot{E} = \int_A \left(\tfrac{1}{2} \rho q_\perp^2 + p_{CG} + \rho g z_G \right) q_\perp \, dA =$$

$$\int_A \tfrac{1}{2} \rho q_\perp^3 \, dA + \int_A \left(p_{CG} + \rho g z_{CG} \right) q_\perp \, dA =$$

$$K_e \tfrac{1}{2} \rho V^3 A + \left(p_{CG} + \rho g z_{CC} \right) V A =$$

$$Q \left[K_e \tfrac{1}{2} \rho V^2 + p_{CG} + \rho g z_{CG} \right] =$$

$$\rho g Q H_e \quad [Nm/s = \text{Watts} = \text{units of power}]$$

where

$$Q = \int_A q_\perp \, dA = \text{Discharge} = VA \;;\; V = Q/A$$

$$K_e = \text{energy coefficient} = \frac{\int_A q_\perp^3 \, dA}{V^3 A} \quad (\approx 1 \text{ if well behaved flow})$$

$$H_e = K_e \frac{V^2}{2g} + \frac{p_{CG}}{\rho g} + z_{CG} \approx H = \frac{V^2}{2g} + \frac{p_{CG}}{\rho g} + z_{CG}$$