1.206J/16.77J/ESD.215J
Airline Schedule Planning

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1.206J/16.77J/ESD.215J Airline Schedule Planning: Multi-commodity Flows

Outline

- Applications
- Problem Definition
- Formulations
- Solutions
- Results

Application I

- Package flow problem (express package delivery operation)
  - Shipments have specific origins and destinations and must be routed over a transportation network
  - Each set of packages with a common origin-destination pair is called a commodity
    - Time windows (availability and delivery time) associated with packages
  - The objective might be to minimize total costs, find a feasible flow, ...

Application II

- Passenger mix problem
  - Given a fixed schedule of flights, a fixed fleet assignment and a set of customer demands for air travel service on this fleeted schedule, the airline's objective is to maximize revenues by accommodating as many high fare passengers as possible
  - For some flights, demand exceeds seat supply and passengers must be *spilled* to other itineraries of either the same or another airline

Application III

- Message routing problem
  - In a telecommunications or computer network, requirements exist for transmission lines and message requests, or commodities.
  - The problem is to route the messages from their origins to their respective destinations at minimum cost

MCF Networks

- Set of nodes
  - Each node associated with the supply of or demand for commodities
- Set of arcs
  - Cost per unit commodity flow
  - Capacity limiting the *total* flow of all commodities and/ or the flow of individual commodities

MCF Commodity Definitions

- A commodity may originate at a subset of nodes in the network and be destined for another subset of nodes
- A commodity may originate at a single node and be destined for a subset of the nodes
- A commodity may originate at a single node and be destined for a single node

MCF Objectives

- Flow the commodities through the networks from their respective origins to their respective destinations at minimum cost
  – Expressed as distance, money, time, etc.

- Ahuja, Magnanti and Orlin (1993)-- survey of multi-commodity flow models and solution procedures

MCF Problem Formulations -- Linear Programs

- Network flow problems
  - Capacity constraints limit flow of individual commodities
  - Conservation of flow constraints ensure flow balance for individual commodities
  - Flow non-negativity constraints
- With side constraints
  - Bundle constraints restrict total flow of *ALL* commodities on an arc

MCF Constraint Matrix

Network flow problem, commodity k=1

Network flow problem, commodity k=2

Network flow problem, commodity k=3

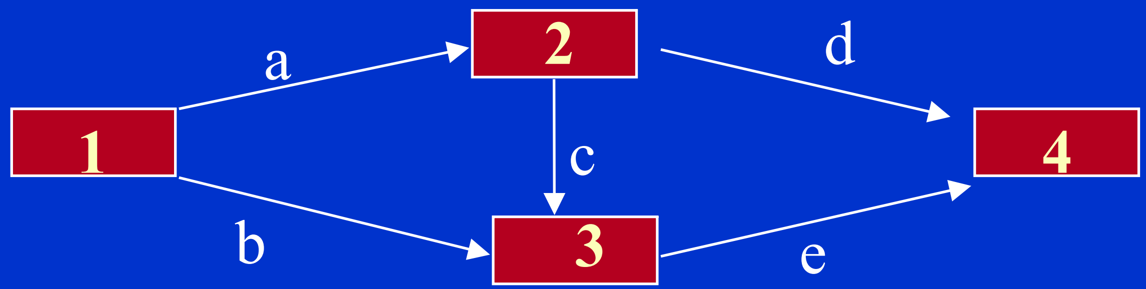
Network flow problem, commodity k=4

Bundle constraints limiting total flow of all commodities to arc capacities

Alternative Formulations for O-D Commodity Case

- Node-Arc Formulation
  - Decision variables: flow of commodity *k* on each arc *ij*
- Path Formulation
  - Decision variables: flow of commodity *k* on each path for *k*
- "Tree" or "Sub-network" Formulation
  - Define: super commodity: set of all (O-D) commodities with the same origin *o* (or destination *d*)
  - Decision variables: quantity of the super commodity *k'* assigned to each "tree" or "sub-network" for *k'*
- Formulations are equivalent

Sample Network



Arcs

i	j	cost	capy
1	2	1	20
1	3	2	10
2	3	3	20
2	4	4	10
3	4	5	40

Commodities

#	o	d	quant
1	1	3	5
2	1	4	15
3	2	4	5
4	3	4	10

Notation

Parameters

- A: set of all network arcs
- K: set of all commodities
- N: set of all network nodes
- O(k) [D(k)]: origin [destination] node for commodity k
- c_{ij}^k : per unit cost of commodity k on arc ij
- u_{ij} : total capacity on arc ij (assume u_{ij}^k is unlimited for each k and each ij)
- d_k : total quantity of commodity k

Decision Variables

- x_{ij}^k : number of units of commodity k assigned to arc ij

Node-Arc Formulation

Minimize $\sum_{ij}\sum_{k} c_{ij}^{k} x_{ij}^{k}$

subject to

$$\sum_{j} x_{ij}^{k} - \sum_{j} x_{ji}^{k} = d_k \quad \text{if } \; i \in O(k)$$

$$= -d_k \quad \text{if } \; i \in D(k) \qquad \text{: Conservation of Flow}$$

$$= 0 \qquad otherwise$$

$$\sum_{k} x_{ij}^{k} \le u_{ij} \qquad \forall (i,j) \in A \qquad \text{: Bundle constraints}$$

$$x_{ij}^{k} \ge 0 \quad \forall (i,j) \in A, k \in K \qquad \text{: Nonnegativity constraints}$$

	k 1					*k 2*					*k 3*					*k 4*					
	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	RHS
1	1	1																			$= d_1$
2	-1		1	1																	$= 0$
3		-1	-1		1																$= -d_1$
4				-1	-1																$= 0$
1						1	1														$= d_2$
2						-1		1	1												$= 0$
3							-1	-1		1											$= 0$
4									-1	-1											$= -d_2$
1											1	1									$= 0$
2											-1		1	1							$= d_3$
3												-1	-1		1						$= 0$
4														-1	-1						$= -d_3$
1																1	1				$= 0$
2																-1		1	1		$= 0$
3																	-1	-1		1	$= d_4$
4																			-1	-1	$= -d_4$
a	1					1					1					1					$\le u_a$
b		1					1					1					1				$\le u_b$
c			1					1					1					1			$\le u_c$
d				1					1					1					1		$\le u_d$
e					1					1					1					1	$\le u_e$
	c_a^1 x_a^1	c_b^1 x_b^1	c_c^1 x_c^1	c_d^1 x_d^1	c_e^1 x_e^1	c_a^2 x_a^2	c_b^2 x_b^2	c_c^2 x_c^2	c_d^2 x_d^2	c_e^2 x_e^2	c_a^3 x_a^3	c_b^3 x_b^3	c_c^3 x_c^3	c_d^3 x_d^3	c_e^3 x_e^3	c_a^4 x_a^4	c_b^4 x_b^4	c_c^4 x_c^4	c_d^4 x_d^4	c_e^4 x_e^4	

Additional Notation

Parameters

- P^k: set of all paths for commodity k, for all k
- c_p : per unit cost of commodity k on path p $= \Sigma_{ij \in p} c_{ij}^k$
- δ_{ij}^p : $= 1$ if path p contains arc ij; and $= 0$ otherwise

Decision Variables

- f_p: fraction of total quantity of commodity k assigned to path p

O/D Based Path Formulation

Minimize $\sum_{k} \sum_{p \in P^k} d_k C_p f_p$

subject to

$$\sum_{k} \sum_{p \in P^k} d_k f_p \delta_{ij}^p \le u_{ij} \quad \forall (i,j) \in A$$: Bundle constraints

$$\sum_{p \in P^k} f_p = 1 \quad \forall k \in K$$: Flow balance constraints

$$f_p \ge 0 \quad \forall p \in P^k, k \in K$$: Non-neg. constraints

	Path									
	k=1		k=2			k=3		k=4	RHS	Dual
a	d_1	0	d_2	d_2	0	0	0	0	<= u_a	$-\pi_a$
b	0	d_1	0	0	d_2	0	0	0	<= u_b	$-\pi_b$
c	d_1	0	d_2	0	0	d_3	0	0	<= u_c	$-\pi_c$
d	0	0	0	d_2	0	0	d_3	0	<= u_d	$-\pi_d$
e	0	0	d_2	0	d_2	d_3	0	d_4	<= u_e	$-\pi_e$
k=1	1	1							= 1	σ^1
k=2			1	1	1				= 1	σ^2
k=3						1	1		= 1	σ^3
k=4								1	= 1	σ^4
Cost.	$C_1 d_1$	$C_2 d_1$	$C_3 d_2$	$C_4 d_2$	$C_5 d_2$	$C_6 d_3$	$C_7 d_3$	$C_8 d_3$		
Variable	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8		

Additional Notation

Parameters

- S: set of source nodes $n \in N$ for all commodities
- Q^s: the set of all sub-networks originating at s
- TC_q^s: total cost of sub-network q originating at s
- ζ_p^q : = 1 if path p is contained in sub-network q; and = 0 otherwise

Decision Variables

- R_q^s : fraction of total quantity of the super commodity originating at s assigned to sub-network q

Sub-network Formulation

$$\text{Minimize} \quad \sum_{s \in S} \sum_{q \in Q^s} \Big(\sum_{k \in q} \sum_{p \in P^k} C_p \, \zeta_p^{\,q} \, d_k \Big) R_q^s$$

subject to

$$\sum_{s} \sum_{q \in Q^s} \Big(\sum_{k \in s} \sum_{p \in P^k} d_k \, \delta_{ij}^p \Big) R_q^s \zeta_p^q \le u_{ij} \quad \forall (i, j) \in A$$: Capacity Limits on Each Arc

$$\sum_{q \in Q^s} R_q^s = 1 \qquad \forall s \in S$$: Mass Balance Requirements

$$R_q^s \ge 0 \qquad \forall q \in Q^s, s \in S$$: Nonnegative Path Flow Variables

	Sub- network										
	o=1						o=2		o=3	RHS	Dual
a	d_1+d_2	d_1+d_2	d_1	d_2	d_2	0	0	0	0	<= u_a	π_a
b	0	0	d_2	d_1	d_1	d_1+d_2	0	0	0	<= u_b	π_b
c	d_1	d_1+d_2	d_1	0	d_2	0	d_3	0	0	<= u_c	π_c
d	d_2	0	0	d_2	0	0	0	d_3	0	<= u_d	π_d
e	0	d_2	d_2	0	d_2	d_2	d_3	0	d_4	<= u_e	π_e
o=1	1	1	1	1	1	1				= 1	σ^1
o=2							1	1		= 1	σ^2
o=3									1	= 1	σ^3
Cost.	TC_1^1	TC_2^1	TC_3^1	TC_4^1	TC_5^1	TC_6^1	TC_1^2	TC_2^2	TC_1^3		
Variable	R_1^1	R_2^1	R_3^1	R_4^1	R_5^1	R_6^1	R_1^2	R_2^2	R_1^3		

Linear MCF Problem Solution

- Obvious Solution
  – LP Solver
- Difficulty
  – Problem Size: (|N|=|Nodes|, |C|=|Commodities|, |A|=|Arcs|)
    - Node-arc formulation:
      – Constraints: |N|*|C| + |A|
      – Variables: |A|*|C|
    - Path formulation:
      – Constraints: |A| + |C|
      – Variables: |Paths for *ALL* commodities|
    - Sub-network formulation:
      – Constraints: |A|+|Origins|
      – Variables: |Combinations of Paths by Origin|

General MCF Solution Strategy

- Try to Decompose a Hard Problem Into a Set of Easy Problems

MCF Solution Procedures I

- Partitioning Methods
  - Exploit Network Structure to Speed Up Simplex Matrix Computations
- Resource-Directive Decomposition
  - Repeat until Optimal:
    - Allocate Arc Capacity Among Commodities
    - Find Optimal Flows Given Allocation

MCF Solution Procedures II

- Price-Directive Decomposition
  - Repeat until Optimal:
    - Modify Flow Cost on Arc
    - Ignore Bundle Constraints, Find Optimal Flows

Revisiting the Path Formulation

$$MINIMIZE \ \sum_{k \in K} \sum_{p \in P^k} d_k c_p f_p$$

subject to: $$\sum_{p \in P^k} \sum_{k \in K} d_k f_p \delta_{ij}^p \leq u_{ij} \quad \forall ij \in A$$

$$\sum_{p \in P(k)} f_p = 1 \quad \forall k \in K$$

$$f_p \geq 0 \quad \forall p \in P^k, \ \forall k \in K$$

By-products of the Simplex Algorithm: Dual Variable Values

Duals

$-\pi_{ij}$: the dual variable associated with the bundle constraint for arc ij (π is non-negative)

σ^k : the dual variable associated with the commodity constraints

Economic Interpretation

π_{ij} : the value of an additional unit of capacity on arc ij

σ^k/d_k : the minimal cost to send an additional unit of commodity k through the network

Modified Costs

<u>Definition:</u> Modified cost for arc *ij and* commodity $k = c_{ij}^{k} + \pi_{ij}$

<u>Definition:</u> Modified cost for path p and commodity $k = \Sigma_{ij \in A} (c_{ij}^{k} + \pi_{ij})\delta_{ij}^{p}$

Optimality Conditions for the Path Formulation

f^*_p and π^*_{ij}, σ^{*k} are optimal for all k and all ij iff:

Primal feasibility is satisfied

1. $\Sigma_{p \in P^k} \Sigma_{k \in K} \mathrm{d}_k f^*_p \delta_{ij}^p \leq u_{ij} \quad \forall\, ij \in A$
2. $\Sigma_{p \in P(k)} f^*_p = 1 \quad \forall\, k \in K$
3. $f^*_p \geq 0 \;\; \forall\, p \in P^k, \; \forall\, k \in K$

Complementary slackness is satisfied

1. $\pi^*_{\mathrm{ij}}(\Sigma_{p \in P^k} \Sigma_{k \in K} \mathrm{d}_k f^*_p \delta_{ij}^p - u_{ij}) = 0, \quad \forall\, ij \in A$
2. $\sigma^{*k} (\Sigma_{p \in P^k} f^*_p - 1) = 0, \quad \forall\, k \in K$

Dual feasibility is satisfied (reduced cost is non-negative for a minimization problem)

1. $(\mathrm{d}_k c_p + \Sigma_{ij \in A} \mathrm{d}_k \pi_{ij} \delta_{ij}^p) - \sigma^k = \mathrm{d}_k (\Sigma_{ij \in A} (c_{ij}^k + \pi_{ij}) \delta_{ij}^p - \sigma^k / \mathrm{d}_k) \geq 0, \; \forall\, p \in P^k, \; \forall\, k \in K$

Multi-commodity Flow Optimality Conditions

- **The price for an additional unit of capacity is 0 unless capacity is fully utilized**

  1. $\pi^*_{ij}(\Sigma_{p \in P^k} \Sigma_{k \in K} d_k f^*_p \delta_{ij}^p - u_{ij}) = 0, \ \forall ij \in A$

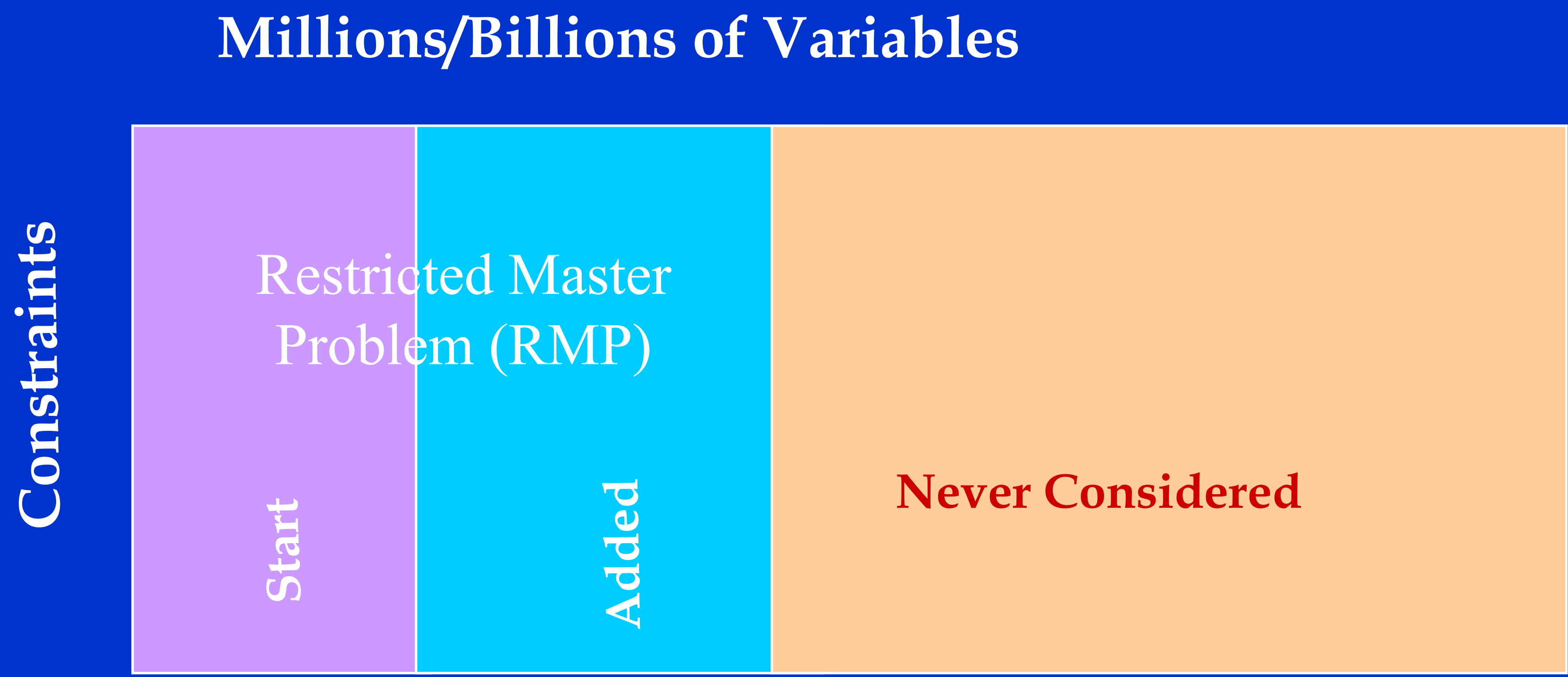
- **A path *p* for commodity *k* is utilized only if its "modified cost" (that is, $\Sigma_{ij \in A} (c_{ij}^k + \pi^*_{ij} \delta_{ij}^p)$) is minimal, for all paths $p \in P^k$**

  1. Reduced Costs all non-negative:

  $$c'_p = d_k (\Sigma_{ij \in A} (c_{ij}^k + \pi^*_{ij}) \delta_{ij}^p - \sigma^{*k} / d_k) \geq 0, \quad \forall p \in P^k, \ \forall k \in K$$

  2. $f^*_p (\Sigma_{ij \in A} (c_{ij}^k + \pi^*_{ij}) \delta_{ij}^p - \sigma^{*k} / d_k) = 0, \quad \forall p \in P^k, \ \forall k \in K$

Column Generation- A Price Directive Decomposition



RMP and Optimality Conditions

Consider f^*_p and π^*_{ij}, σ^{*k} optimal for RMP, then

Primal feasibility is satisfied

1. $\Sigma_{p \in P^k} \Sigma_{k \in K} d_k f^*_p \delta_{ij}^p \leq u_{ij} \quad \forall ij \in A$
2. $\Sigma_{p \in P(k)} f^*_p = 1 \quad \forall k \in K$
3. $f^*_p \geq 0 \;\; \forall p \in P^k, \; \forall k \in K$

Complementary slackness is satisfied

1. $\pi^*_{ij}(\Sigma_{p \in P^k} \Sigma_{k \in K} d_k f^*_p \delta_{ij}^p - u_{ij}) = 0, \quad \forall ij \in A$
2. $\sigma^{*k} (\Sigma_{p \in P^k} f^*_p - 1) = 0, \quad \forall k \in K$

Dual feasibility is guaranteed (reduced cost is non-negative) ONLY for a path p included in RMP

1. $(d_k c_p + \Sigma_{ij \in A} d_k \pi_{ij} \delta_{ij}^p) - \sigma^k = d_k (\Sigma_{ij \in A} (c_{ij}^k + \pi_{ij}) \delta_{ij}^p - \sigma^k / d_k) \geq 0, \; \forall p \in P^k, \; \forall k \in K$

LP Solution: Column Generation

- Step 1: Solve *Restricted Master Problem* (RMP) with subset of all variables (columns)
- Step 2: Solve *Pricing Problem* to determine if any variables when added to the RMP can improve the objective function value (that is, if any variables have negative reduced cost)
- Step 3: If variables are identified in Step 2, add them to the RMP and return to Step 1; otherwise STOP

Pricing Problem

- Given π, the optimal (non-negative) duals for the current restricted master problem, the pricing problem, for each $p \in P^k$, $k \in K$ is

$$\min_{p \in P^k} (d_k (\Sigma_{ij \in A} (c_{ij}^k + \pi_{ij}) \delta_{ij}^p - \sigma^k / d_k)$$

Or, equivalently:

$$\min_{p \in P^k} \Sigma_{ij \in A} (c_{ij}^k + \pi_{ij}) \delta_{ij}^p$$

➢ *A shortest path problem for commodity k (with modified arc costs)*

Example- Iteration 1

	Path								RHS	Dual
	k=1		k=2			k=3		k=4	RHS	Dual
a	5	0	15	15	0	0	0	0	<= 20	$\pi_a = 0$
b	0	5	0	0	15	0	0	0	<= 10	$\pi_b = 0$
c	5	0	15	0	0	5	0	0	<= 20	$\pi_c = 0$
d	0	0	0	15	0	0	5	0	<= 10	$\pi_d = 0$
e	0	0	15	0	15	5	0	10	<= 40	$\pi_e = 0$
k=1	1	1							= 1	$\sigma^1 = 10$
k=2			1	1	1				= 1	$\sigma^2 = 135$
k=3						1	1		= 1	$\sigma^3 = 20$
k=4								1	= 1	$\sigma^4 = 50$
Cost.	20	10	135	75	105	40	20	50		
Variable	f_1	f_2 =1	f_3 =1	f_4	f_5	f_6	f_7 =1	f_8 =1		

Example- Iteration 2

	Path									
	k=1		k=2			k=3		k=4	RHS	Dual
a	5	0	15	15	0	0	0	0	<= 20	$\pi_a = 0$
b	0	5	0	0	15	0	0	0	<= 10	$\pi_b = 2$
c	5	0	15	0	0	5	0	0	<= 20	$\pi_c = 0$
d	0	0	0	15	0	0	5	0	<= 10	$\pi_d = 4$
e	0	0	15	0	15	5	0	10	<= 40	$\pi_e = 0$
k=1	1	1							= 1	$\sigma^1 = 20$
k=2			1	1	1				= 1	$\sigma^2 = 135$
k=3						1	1		= 1	$\sigma^3 = 40$
k=4								1	= 1	$\sigma^4 = 50$
Cost.	20	10	135	75	105	40	20	50		
Variable	f_1	f_2 =1	f_3 =1/3	f_4 = 1/3	f_5 =1/3	f_6	f_7 =1	f_8 =1		

MCF Optimality Conditions

- For each $p \in P^k$, for each k, the reduced cost c'_p:
  - $c'_p = (d_k c_p + \Sigma_{ij \in A} d_k \pi_{ij} \delta_{ij}^p) - \sigma^k = \Sigma_{ij} (d_k c_{ij}^k + d_k \pi_{ij}) \delta_{ij}^p - \sigma^k = \Sigma_{ij} (c_{ij}^k + \pi_{ij}) \delta_{ij}^p - \sigma^k / d_k \geq 0$
    - where π, σ are the optimal duals for the current restricted master problem
  - $c'_p = 0$, for each utilized path p implies

    $\Sigma_{ij} (d_k c_{ij}^k + d_k \pi_{ij}) \delta_{ij}^p = \sigma^k$

    or equivalently,

    $\Sigma_{ij} (c_{ij}^k + \pi_{ij}) \delta_{ij}^p = \sigma^k / d_k$
  - So if, $min_{p \in P(k)} c'_p = \Sigma_{ij} (c_{ij}^k + \pi_{ij}) \delta_{ij}^{p*} - \sigma^k / d_k \geq 0$, the current solution to the restricted master problem is optimal for the original problem
  - If $min_{p \in P(k)} c'_p = \Sigma_{ij} (c_{ij}^k + \pi_{ij}) \delta_{ij}^{p*} - \sigma^k / d_k < 0$, add p^* to restricted master problem

Data Set

- Data Set

Nodes		807
Links		1,363
	capacitated	292
	uncapacitated	1,071
O/D		17,539
	# Origin	136

- Constraint Matrix Size

			Improvement
	row	column	new_row
Node_Arc	14,155,336	23,905,657	-
Path	18,902	-	17,832
Sub-network	1,499	-	428

Computational Results

- Number of Nodes: 807
- Number of Links: 1,363
- Number of Commodities: 17,539
- Computational Result (IBM RS6000, Model 370)
  – Path Model: 44 minutes
  – Sub-network Model: < 1 minute

Conclusions I

- Choose your formulation carefully
  - Trade-off memory requirements and solution time
  - Sub-network formulation can be effective when low level of congestion in the network
- Problem size often mandates use of combined column and row generation

Conclusions II

- Solution time is affected dramatically by
  - The complexity of the pricing problem
  - Exploitation of problem structure, pre-processing, LP solver selection, etc.