1.206J/16.77J/ESD.215J
Airline Schedule Planning

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1.206J/16.77J/ESD.215J Multi-commodity Network Flows: A Keypath Formulation

- Outline
  - Path formulation for multi-commodity flow problems revisited
  - Keypath formulation
  - Example
  - Keypath solution algorithm
    - Column generation
    - Row generation

Path Notation

Sets

A: set of all network arcs

K: set of all commodities

N: set of all network nodes

Parameters

u_{ij} : total capacity on arc ij

d_k : total quantity of commodity k

P^k: set of all paths for commodity k, for all k

Parameters (cont.)

c_p : per unit cost of commodity k on path p = $\Sigma_{ij \in p}\, c_{ij}^k$

δ_{ij}^p : = 1 if path p contains arc ij; and = 0 otherwise

Decision Variables

f_p: fraction of total quantity of commodity k assigned to path p

The Path Formulation Revisited

$$MINIMIZE \ \Sigma_{k \in K} \Sigma_{p \in P^k} \mathrm{d}_k c_p f_p$$

subject to: $$\Sigma_{p \in P^k} \Sigma_{k \in K} d_k f_p \delta_{ij}^p \leq u_{ij} \quad \forall \, ij \in A$$

$$\Sigma_{p \in P^k} f_p = 1 \quad \forall \, k \in K$$

$$f_p \geq 0 \ \forall \, p \in P^k, \ \forall \, k \in K$$

The Keypath Concept

- The path formulation for MCF problems can be recast equivalently as follows:
  - Assign all flow of commodity k to a selected path p, called the *keypath*, for each commodity $k \in K$
    - Often the keypath is the minimum cost path for k
    - The resulting flow assignment is often infeasible
      - One or more arc capacity constraints are violated
    - If the resulting flows are feasible and the keypaths are minimum cost, the flow assignment is optimal
  - Solve a linear programming formulation to minimize the cost of adjusting flows to achieve feasibility
    - Flow adjustments involve removing flow of k from its keypath p and placing it on alternative path $p' \in P^k$, for each $k \in K$

Additional Keypath Notation

Parameters

$p(k)$: keypath for commodity k

Q_{ij} : total initial (flow assigned to keypaths) on arc ij
$= \Sigma_{k \in K} d_k \delta_{ij}^{p(k)}$

$c^r_{p(k)}$: $= c_r - c_{p(k)} = \Sigma_{ij \in A} c_{ij} \delta_{ij}^{r} - \Sigma_{ij \in A} c_{ij} \delta_{ij}^{p(k)}$;
change in cost when one unit of commodity k is shifted from keypath *p(k)* to path *r* (Note: typically non-negative if *p(k)* has minimum cost)

Decision Variables

$f^r_{p(k)}$: fraction of total quantity of commodity k removed from keypath *p(k)* to path *r*

The Keypath Formulation

$$\text{Min} \quad \sum_{k \in K} \sum_{r \in P^k} \left(c_{p(k)}{}^{r} \right) d_k f_{p(k)}^{r}$$

s.t.:

$$-\sum_{k \in K} \sum_{r \in P^k} \delta_{ij}^{p(k)} d_k f_{p(k)}^{r} + \sum_{k \in K} \sum_{r \in P^k} \delta_{ij}^{r} d_k f_{p(k)}^{r}$$

$$\leq u_{ij} - Q_{ij} \qquad \forall ij \in A$$

$$\sum_{r \in P^k} f_{p(k)}^{r} \leq 1 \qquad \forall k \in K$$

$$f_{p(k)}^{r} \geq 0 \qquad \forall r \in P^k \quad \forall k \in K$$

Associated Dual Variables

Duals

- π_{ij} : the dual variable associated with the bundle constraint for arc ij (π is non-negative)
- σ^{k}: the dual variable associated with the commodity constraints (σ is non-negative)

Economic Interpretation

π_{ij} : the value of an additional unit of capacity on arc ij

σ^{k}/d_k : the minimal cost to remove an additional unit of commodity *k* from its keypath and place on another path

Optimality Conditions for the Path Formulation

f^*_p and π^*_{ij}, σ^{*k} are optimal for all k and all ij if:

1. Primal feasibility is satisfied
2. Complementary slackness is satisfied
3. Dual feasibility is satisfied (reduced cost is non-negative for a minimization problem)

Modified Costs

<u>Definition:</u> Reduced cost for path *r*, commodity *k*

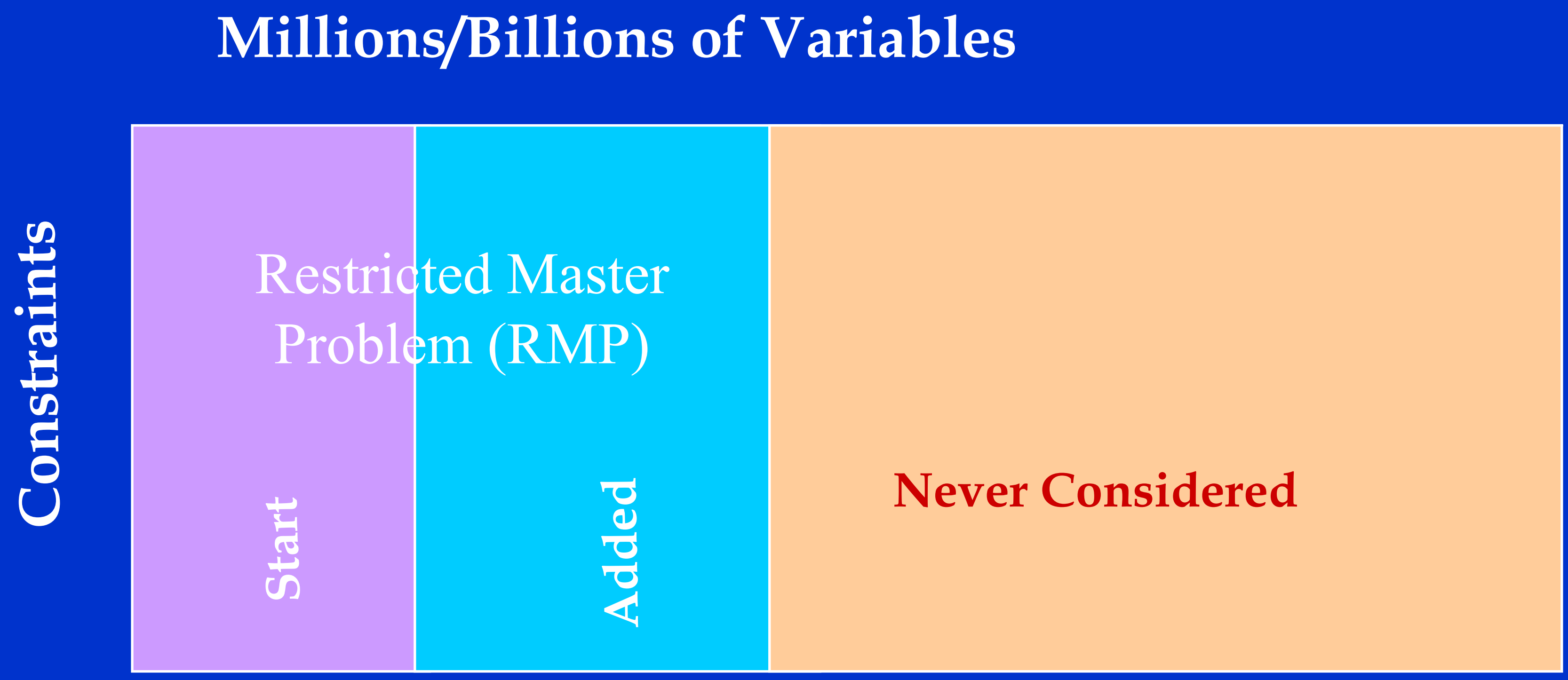
$$= \Sigma_{ij \in A} c_{ij}^{k} d_k \delta_{ij}^{r} - \Sigma_{ij \in A} c_{ij}^{k} d_k \delta_{ij}^{p(k)} + \Sigma_{ij \in A} \pi_{ij} d_k \delta_{ij}^{r} - \Sigma_{ij \in A} \pi_{ij} d_k \delta_{ij}^{p(k)} + \sigma^{k}$$

$$= \Sigma_{ij \in A} (c_{ij}^{k} + \pi_{ij}) \delta_{ij}^{r} - \Sigma_{ij \in A} (c_{ij}^{k} + \pi_{ij}) \delta_{ij}^{p(k)} + \sigma^{k} / d_k$$

<u>Definition:</u> Let modified cost for arc *ij and* commodity *k* $= c_{ij}^{k} + \pi_{ij}$

- **Reduced cost is non-negative for all commodity *k* variables if the modified cost of path *r* equals or exceeds the modified cost of *p(k)* less σ^{k}/d_k**

Column Generation- A Price Directive Decomposition



LP Solution: Column Generation

- Step 1: Solve *Restricted Master Problem* (RMP) with subset of all variables (columns)
- Step 2: Solve *Pricing Problem* to determine if any variables when added to the RMP can improve the objective function value (that is, if any variables have negative reduced cost)
- Step 3: If variables are identified in Step 2, add them to the RMP and return to Step 1; otherwise STOP

Pricing Problem

- Given π and σ^k, the optimal (non-negative) duals for the current restricted master problem and the keypath $p(k)$, the pricing problem, for each $k \in K$ is

$$\min_{r \in P^k} \left(d_k \left(\Sigma_{ij \in A} (c_{ij}^k + \pi_{ij})\, \delta_{ij}^r - \Sigma_{ij \in A} (c_{ij}^k + \pi_{ij})\, \delta_{ij}^{p(k)} + \sigma^k / d_k\right)\right)$$

Or, letting $C = \Sigma_{ij \in A} (c_{ij}^k + \pi_{ij})\, \delta_{ij}^{p(k)} - \sigma^k / d_k$ equivalently:

$$\min_{r \in P^k} \Sigma_{ij \in A} (c_{ij}^k + \pi_{ij})\, \delta_{ij}^r - C$$

➢ A shortest path problem for commodity k (with modified arc costs). If $\min_{r \in P^k} \Sigma_{ij \in A} (c_{ij}^k + \pi_{ij})\, \delta_{ij}^r - C \geq 0$, then the original problem is solved, else add column corresponding to $x_{p(k)}^r$ to the master problem

Example- Iteration 1

Let $p(1) = 2$; $p(2) = 4$; $p(3) = 7$; $p(4) = 8$ (** denotes keypath)

	Path									
	k=1		k=2			k=3		k=4	RHS	Dual
a	5	0	15-15	15-15	0-15	0	0	0	<= 20-15	$\pi_a = 0$
b	-5	5-5	0	0	15	0	0	0	<= 10-5	$\pi_b = 0$
c	5	0	15	0	0	5	0	0	<= 20-0	$\pi_c = 0$
d	0	0	0-15	15-15	0-15	0-5	5-5	0	<= 10-20	$\pi_d = 2$
e	0	0	15	0	15	5	0	10-10	<= 40-10	$\pi_e = 0$
k=1	1	1 -1							<= 1	σ^1
k=2			1	1-1	1				<= 1	σ^2
k=3						1	1-1		<= 1	σ^3
k=4								1-1	<= 1	σ^4
Cost.	20-10	10-10	165-75	75-75	135-75	40-30	30-30	50-50		
Variable	$f_2^1 = 0$	f_2^2 **	f_4^3	f_4^4 **	$f_4^5 = 0$	$f_7^6 = 2$	f_7^7 **	f_8^8 **		

NOTE: Gray columns not included in keypath formulation; purple elements are initial keypath matrix

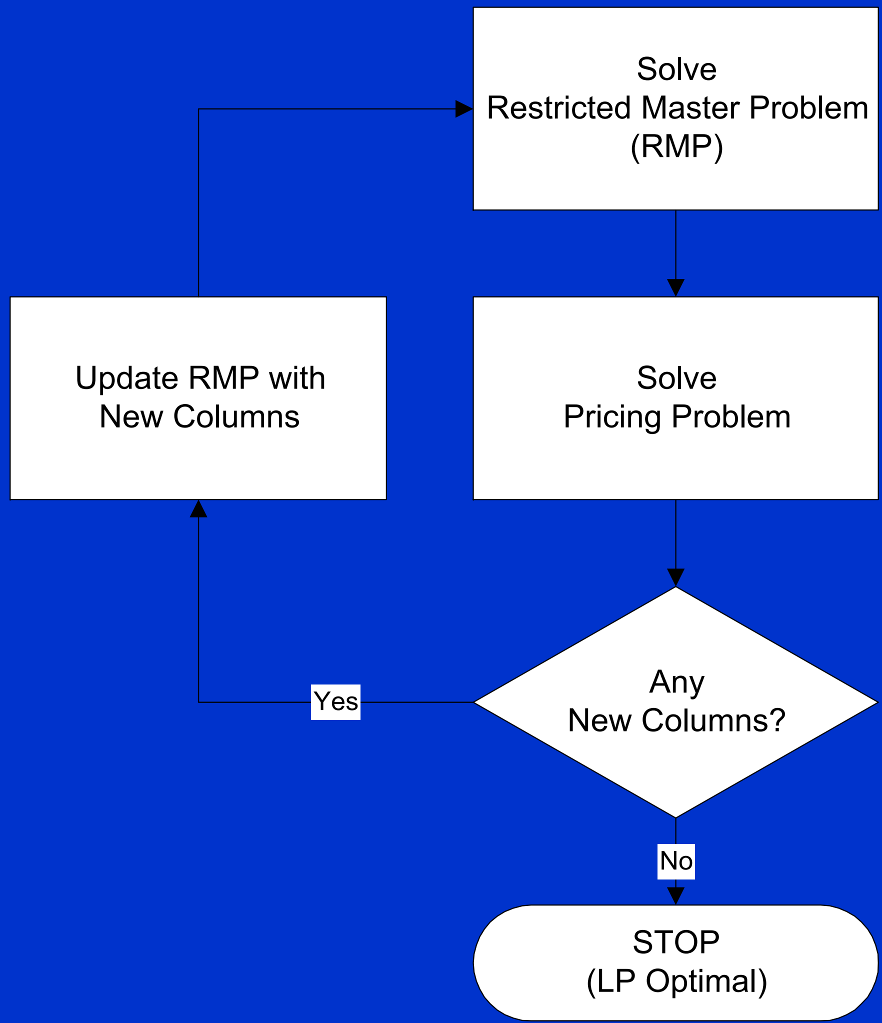
Example- Iteration 2

Let $p(1) = 2$; $p(2) = 4$; $p(3) = 7$; $p(4) = 8$ (** denotes keypath)

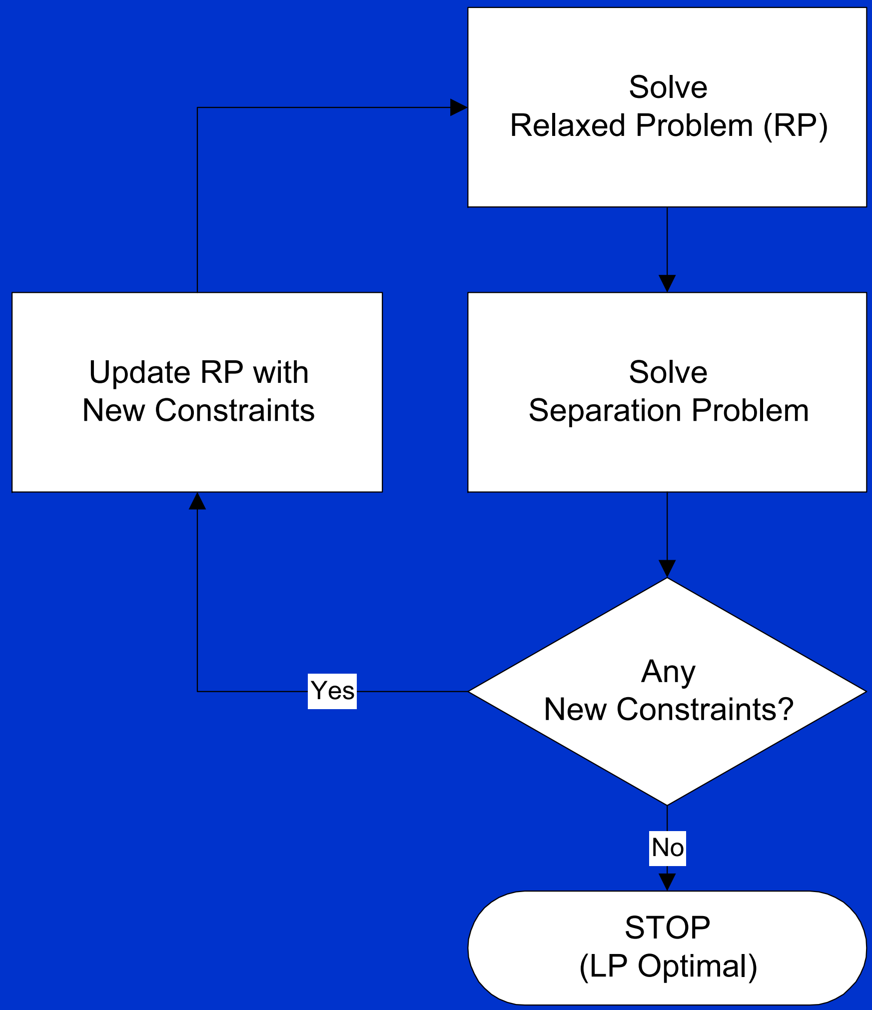
	Path									
	k=1		k=2			k=3		k=4	RHS	Dual
a	5	0	15-15	15-15	0-15	0	0	0	<= 20-15	$\pi_a = 0$
b	-5	5-5	0	0	15	0	0	0	<= 10-5	$\pi_b = 0$
c	5	0	15	0	0	5	0	0	<= 20-0	$\pi_c = 0$
d	0	0	0-15	15-15	0-15	0-5	5-5	0	<= 10-20	$\pi_d = 4$
e	0	0	15	0	15	5	0	10-10	<= 40-10	$\pi_e = 0$
k=1	1	1 -1							<= 1	σ^1
k=2			1	1-1	1				<= 1	σ^2
k=3						1	1-1		<= 1	$\sigma^3 = 10$
k=4								1-1	<= 1	σ^4
Cost.	20-10	10-10	165-75	75-75	135-75	40-30	30-30	50-50		
Variable	$f_2^1 = 0$	f_2^2 **	f_4^3	f_4^4 **	$f_4^5 = 1/3$	$f_7^6 = 1$	f_7^7 **	f_8^8 **		

2nd iteration: no columns price out, one constraint for commodity 3 is violated and added; and the problem is resolved– feasibility and optimality achieved

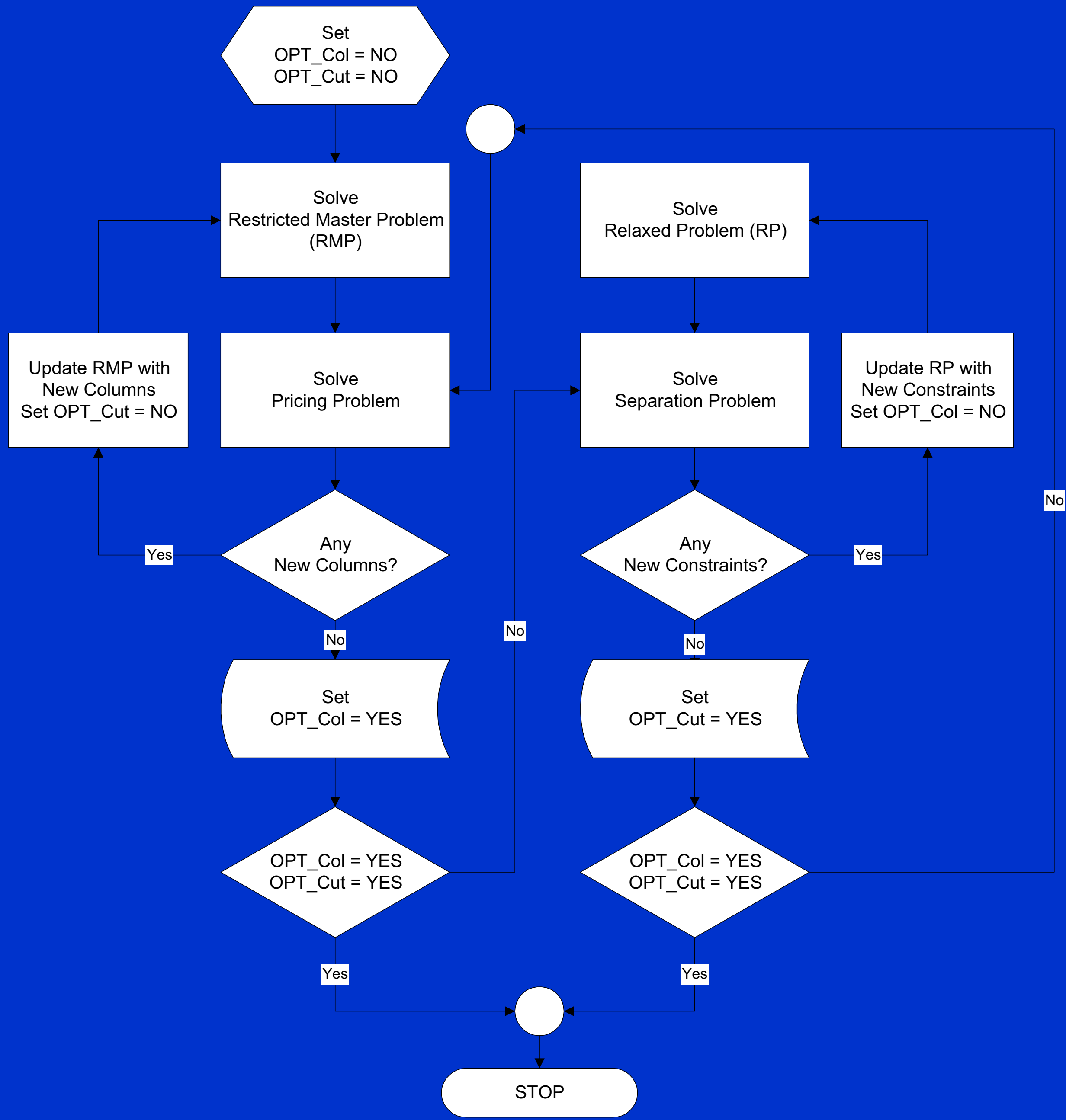
Column Generation



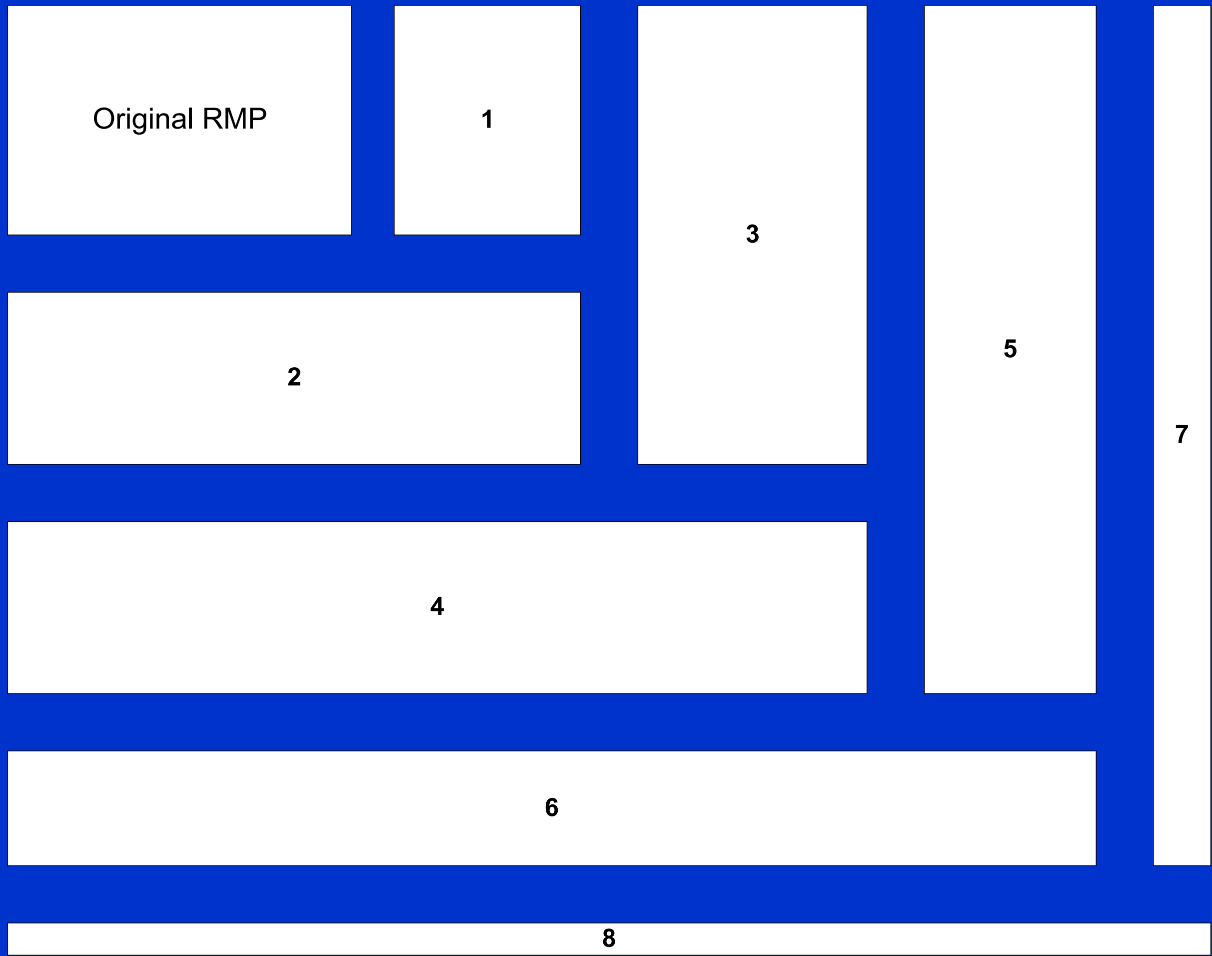
Row Generation



Column and Row Generation



Column and Row Generation: Constraint Matrix



The Benefit of the Keypath Concept

- We are now minimizing the objective function and most of the objective coefficients are __positive__. Therefore, this will guide the decision variables to values of ___0___.
- How does this help?

Solution Procedure

- Use Both Column Generation and Row Generation
- Actual flow of problem
  - Step 1- Define RMP for Iteration 1: Set k =1. Denote an initial subset of columns (A_1) which is to be used.
  - Step 2- Solve RMP for Iteration k: Solve a problem with the subset of columns A_k.
  - Step 3- Generate Rows: Determine if any constraints are violated and express them explicitly in the constraint matrix.
  - Step 4- Generate Columns: Price some of the remaining columns, and add a group (A*) that have a reduced cost less than zero, i.e., $A_{k+1}=[A_k \mid A^*]$
  - Step 5- Test Optimality: If no columns or rows are added, terminate. Otherwise, k =k+1, go to Step 2

Conclusions

- Variable redefinition
  - Allows relaxation of constraints and subsequent (limited) cut generation
  - Does not alter the pricing problem solution
    - Shortest paths with modified costs
  - Allows problems with many commodities, as well as a large underlying network, to be solved with limited memory requirements