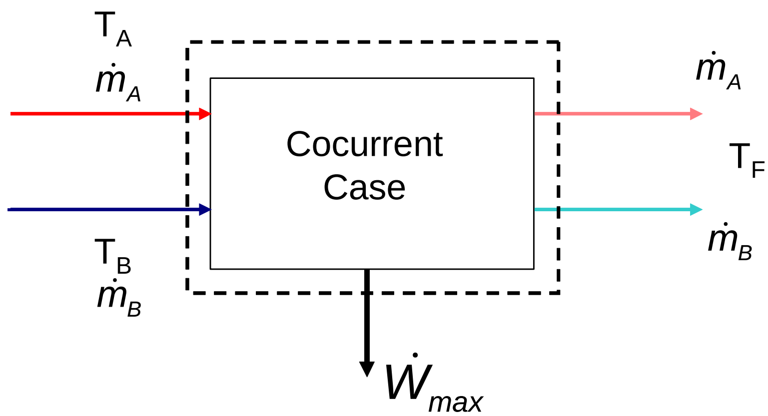
10.40 Thermodynamics **Fall 2003**
Problem Set 3

Problem 4.11 Text – solve part (a) of the problem two different ways:
1.) Using the availability and
2.) without using the availability

Solution:
(a)

Assumptions:
Reversible
C_P = constant



This problem should look very familiar, since it is almost exactly like Problem 4.3 that we did last week. The system for the cocurrent case is depicted in the figure above. It is assumed that the process is reversible. The outlet temperatures of the streams are the same so that the maximum work will be extracted from the streams.

We will determine the maximum work first by using the availability. The system is made up of the two streams entering and exiting the process. For these two streams:

$$\Delta\dot{\underline{B}}_{Total} = \dot{W}_{max} = \Delta\dot{\underline{B}}_A + \Delta\dot{\underline{B}}_B = \left(\Delta\dot{\underline{H}}_A - T_0\Delta\dot{\underline{S}}_A\right) + \left(\Delta\dot{\underline{H}}_B - T_0\Delta\dot{\underline{S}}_B\right) \quad (1)$$

We have assumed that each stream is interacting with some dead state at T_0. This could be the air, some deep ocean reservoir, or anything else imaginable. Since the availability is a state function, only the final end states of streams A and B matter. Substituting for enthalpy and rearranging:

$$\dot{W}_{max} = C_p\left(\dot{m}_A\left(T_F - T_A\right) + \dot{m}_B\left(T_F - T_B\right)\right) + T_0\left(\dot{m}_A\Delta S_A + \dot{m}_B\Delta S_B\right) \quad (2)$$

Performing an entropy balance on the system shows that:

$$\underbrace{d\dot{\underline{S}}}_{=0,SS} = \underbrace{\frac{\delta\dot{Q}}{T}}_{=0} + \dot{m}_A\left(S_{A,in} - S_{A,out}\right) + \dot{m}_B\left(S_{B,in} - S_{B,out}\right)$$
$$0 = \dot{m}_A\left(\Delta S_A\right) + \dot{m}_B\left(\Delta S_B\right) \quad (3)$$

Therefore:

$$\boxed{\dot{W}_{max} = C_p\left(\dot{m}_A\left(T_F - T_A\right) + \dot{m}_B\left(T_F - T_B\right)\right)} \qquad (4)$$

The same result can be found by performing a 1st Law balance around the system shown in the figure above:

$$\underbrace{\Delta\dot{U}}_{=0,SS} = \underbrace{\dot{Q}}_{=0} + \underbrace{\dot{W}}_{\dot{W}_{max}} + \dot{m}_A\left(H_{A,in} - H_{A,out}\right) + \dot{m}_B\left(H_{B,in} - H_{B,out}\right) \qquad (5)$$

$$\boxed{\dot{W}_{max} = C_p\left(\dot{m}_A\left(T_F - T_A\right) + \dot{m}_B\left(T_F - T_B\right)\right)} \qquad (6)$$

(b)

It was already assumed that the pinch point temperature occurred at the outlet of the system. This must be so in order to obtain the maximum work from the system. Now, the derivation of T_F is the same as it was in Problem 4.3. We assume that there are an infinite number of Carnot engines operating between the two streams. For the i^{th} Carnot engine operating between differential sections of streams A and B:

$$\frac{\delta Q_{Ai}}{T_{Ai}} + \frac{\delta Q_{Bi}}{T_{Bi}} = 0 \qquad (7)$$

Performing a 1st Law Balance on the differential part of the hot stream, stream A, that interacts with this i^{th} Carnot engine shows that:

$$\underbrace{\cancel{dU_i}}_{=0,\ SS} = \delta Q_{Ai} + \underbrace{\cancel{\delta W_{Ai}}}_{=0} + (\underbrace{H_{in} - H_{out}}_{\text{=dH, only differential change}})\delta m_A \qquad (8)$$

Dividing by dt, and substituting for dH:

$$\dot{Q}_{Ai} = -\dot{m}_A C_p dT_A \qquad (9)$$

Likewise, a material balance on cold stream, stream B, gives:

$$\dot{Q}_{Bi} = -\dot{m}_B C_p dT_B$$

Notice that our previous assumption that δQ_{Bi} is negative with respect to our Carnot Engine system was correct. Substituting these into equation (2) gives:

$$\frac{\dot{m}_A C_p dT_A}{T_{Ai}} + \frac{\dot{m}_B C_p dT_B}{T_{Bi}} = 0 \qquad (10)$$

Integrating over the infinite number of Carnot engines from the inlet conditions to the exit, where both streams are at the same temperature T_F, and plugging in values gives:

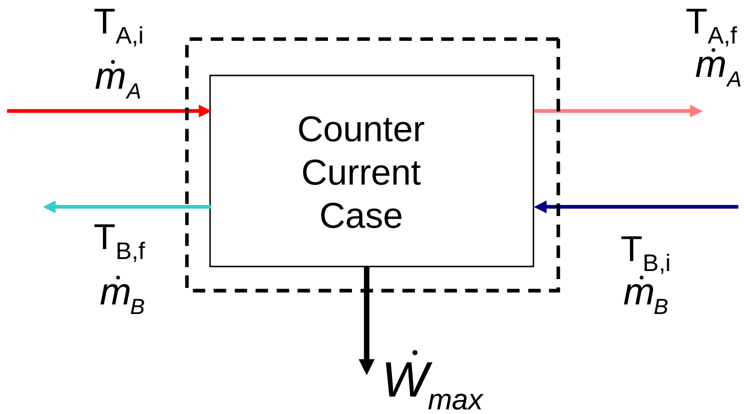
$$\dot{m}_A \int_{T_A}^{T_F} \frac{dT}{T} + \dot{m}_B \int_{T_B}^{T_F} \frac{dT}{T} = 0$$

$$\dot{m}_A \, ln\left(\frac{T_F}{T_A}\right) + \dot{m}_B \, ln\left(\frac{T_F}{T_B}\right) = 0 \qquad (11)$$

$$T_F = \left(T_A^{\dot{m}_A} T_B^{\dot{m}_B}\right)^{\frac{1}{\dot{m}_A + \dot{m}_B}} = \left(T_A T_B^{\frac{\dot{m}_B}{\dot{m}_A}}\right)^{\frac{1}{1+\frac{\dot{m}_B}{\dot{m}_A}}}$$

T_F is the pinch point temperature, since $T_A = T_B = T_F$ at the outlet.

(c)



For the counter current case, it is not immediately intuitive where the pinch point is or what the final temperatures of the outlet streams are, so for now we will leave them both as unknowns. The derivation for the maximum work for the countercurrent case is exactly the same as it is for the cocurrent case. We find that:

$$\dot{W} = C_p\left(\dot{m}_A\left(T_{A,f} - T_{A,i}\right) + \dot{m}_B\left(T_{B,f} - T_{B,i}\right)\right) \tag{12}$$

Performing the same analysis used in part (b):

$$\dot{m}_A\int_{T_{A,i}}^{T_{A,f}}\frac{dT}{T} + \dot{m}_B\int_{T_{B,i}}^{T_{B,f}}\frac{dT}{T} = 0$$

$$\dot{m}_A\, ln\left(\frac{T_{A,f}}{T_{A,i}}\right) + \dot{m}_B\, ln\left(\frac{T_{B,f}}{T_{B,i}}\right) = 0 \tag{13}$$

$$T_{B,f} = T_{B,i}\left(\frac{T_{A,i}}{T_{A,f}}\right)^{\frac{\dot{m}_A}{\dot{m}_B}}$$

Plugging equation (13) into equation (12) gives:

$$\dot{W} = C_p\left(\dot{m}_A\left(T_{A,f} - T_{A,i}\right) + \dot{m}_B\left(T_{B,i}\left(\frac{T_{A,i}}{T_{A,f}}\right)^{\frac{\dot{m}_A}{\dot{m}_B}} - T_{B,i}\right)\right) \tag{14}$$

We now have the work as a function of the outlet temperature only. To find the maximum work, we minimize this function (since $\dot{W}$ is a negative number). To do this, we set the derivative equal to zero:

$$\frac{d\dot{W}}{dT_{A,f}} = 0 = C_p\left(\dot{m}_A + -\left(\frac{\dot{m}_A}{\dot{m}_B}\right)\left(T_{B,i}T_{A,i}^{-\left(\frac{\dot{m}_A}{\dot{m}_B}+1\right)}\right)\right) \tag{15}$$

$$\frac{T_{B,i} T_{A,i}^{\frac{\dot{m}_A}{\dot{m}_B}}}{T_{A,f}^{\left(\frac{\dot{m}_A+\dot{m}_B}{\dot{m}_B}\right)}} = 1 \tag{16}$$

$$T_{A,f} = \left(T_{A,i}^{\dot{m}_A} T_{B,i}^{\dot{m}_B}\right)^{\frac{1}{\dot{m}_A+\dot{m}_B}} = \left(T_{A,i} T_{B,i}^{\frac{\dot{m}_B}{\dot{m}_A}}\right)^{\frac{1}{1+\frac{\dot{m}_B}{\dot{m}_A}}} \tag{17}$$

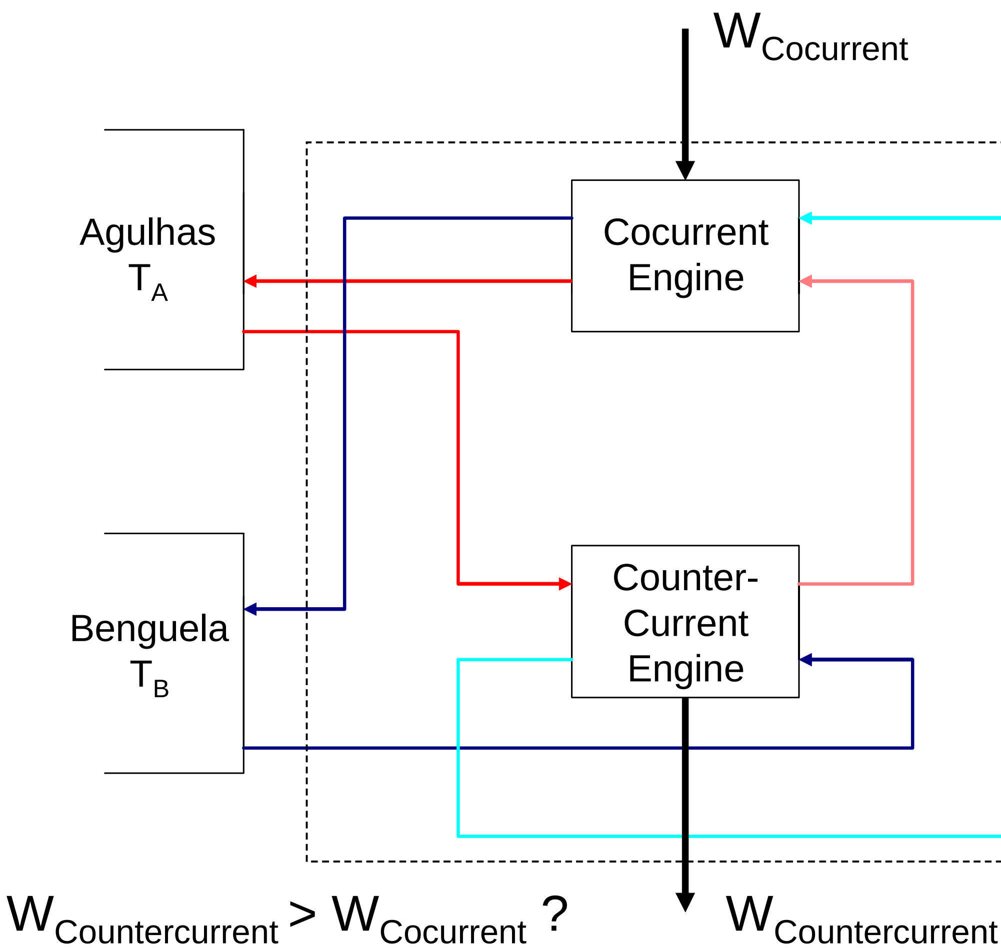
This is exactly the same result we got in for the cocurrent case. The pinch point is at one of the ends of the system (the temperature change is linear due to the assumption of a constant C_p). Which end depends upon the relative mass flow rates of streams A and B. If the flow rate of stream A is larger, the pinch point is at its inlet, and vice versa.

To be sure that this is a minimum, we check that the second derivative is greater than zero:

$$\frac{d^2\dot{W}}{dT_{A,f}^2} = \underbrace{\left(\frac{\dot{m}_A}{\dot{m}_B}+1\right)}_{+} \underbrace{T_{B,i} T_{A,i}^{\frac{\dot{m}_A}{\dot{m}_B}} T_{A,f}^{-\left(\frac{\dot{m}_A}{\dot{m}_B}+2\right)}}_{+} > 0 \tag{18}$$

So we find that the maximum work for the countercurrent case is exactly the same as it is for the cocurrent case, with the outlet streams exiting at the same temperature.

We could have also come to this conclusion by realizing that since availability is a state function, the maximum work we can derive from the two streams is also a state function, so the two outlet streams must be at the same conditions as they were for the cocurrent case. If this were not the case, it would imply that not all the possible work had been extracted from the streams in the cocurrent case. If we had found that the inlet and outlet conditions for the two cases were the same, but we obtained more work (for example) from the countercurrent system than the cocurrent system, it would imply that we could build a perpetual motion machine of the first kind (depicted in the figure below). This is not possible.



(d)
The solution is the same for both cases. Plugging in the numbers, we find
$T_{A,f} = T_{B,f} = T_F = 290$ K
$\dot{W}_{max} = -3.35 \times 10^{13}$ J/s