10.40 Thermodynamics **Fall 2003**
Problem Set 3

Problem 5.26 Text

Solution:

In order to evaluate the derivative, it needs to be in terms that can be found using steam table data. Derivatives at constant pressure or temperature are easiest to evaluate since the steam table is organized by pressure and temperature.

$$v_c^2 = \left(\frac{\partial P}{\partial \rho}\right)_S = \left(\frac{\partial P}{\partial \left(1/V\right)}\right)_S = -V^2\left(\frac{\partial P}{\partial V}\right)_S \tag{1}$$

Chain rule expansion to add T yields,

$$v_c^2 = -V^2 \frac{\left(\frac{\partial P}{\partial T}\right)_S}{\left(\frac{\partial V}{\partial T}\right)_S} \tag{2}$$

Expanding the numerator and denominator by using the triple product and then applying derivative inversion,

$$v_c^2 = -V^2 \frac{\left(\frac{\partial S}{\partial T}\right)_P \left(\frac{\partial P}{\partial S}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_V \left(\frac{\partial V}{\partial S}\right)_T} \tag{3}$$

Applying the definitions of C_p and C_V from Table 3.2 and differentiating H and U, respectively, to transform the derivative definitions,

$$C_P \equiv \left(\frac{\partial H}{\partial T}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_P$$
$$C_V \equiv \left(\frac{\partial U}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \tag{4}$$

Substituting for C_p and C_V in equation (3) yields,

$$v_c^2 = -V^2 \frac{C_p\left(\frac{\partial P}{\partial S}\right)_T}{C_V\left(\frac{\partial V}{\partial S}\right)_T} \tag{5}$$

The next step is to convert the entropy derivatives into pressure, specific volume, and pressure only derivatives by applying Maxwell's reciprocity relationship to the derivative forms of Gibbs free energy and Helmholtz free energy.

$$\left(\frac{\partial}{\partial P}\left(\frac{\partial G}{\partial T}\right)_P\right)_T = -\left(\frac{\partial S}{\partial P}\right)_T = \left(\frac{\partial}{\partial T}\left(\frac{\partial G}{\partial P}\right)_T\right)_P = \left(\frac{\partial V}{\partial T}\right)_P ; \quad \left(\frac{\partial P}{\partial S}\right)_T = -\left(\frac{\partial T}{\partial V}\right)_P$$
$$\left(\frac{\partial}{\partial V}\left(\frac{\partial A}{\partial T}\right)_V\right)_T = -\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial}{\partial T}\left(\frac{\partial A}{\partial V}\right)_T\right)_V = -\left(\frac{\partial P}{\partial T}\right)_V ; \quad \left(\frac{\partial V}{\partial S}\right)_T = \left(\frac{\partial T}{\partial P}\right)_V \tag{6}$$

Insert the relationships from (6) into (5),

$$v_c^2 = V^2 \frac{C_p\left(\frac{\partial T}{\partial V}\right)_P}{C_V\left(\frac{\partial T}{\partial P}\right)_V} \tag{7}$$

Applying the triple product rule to (7) gives,

$$v_c^2 = -V^2 \frac{C_p}{C_V} \frac{1}{\left(\frac{\partial V}{\partial P}\right)_T} = \frac{V\gamma}{\frac{-1}{V}\left(\frac{\partial V}{\partial P}\right)_T} = \frac{V\gamma}{\kappa_T}; \quad \kappa_T \equiv \text{isothermal compressibility} \tag{8}$$

$$\boxed{v_c = \sqrt{\frac{V\gamma}{\kappa_T}}} \tag{9}$$

Evaluate v_c for air and water.

Air

To determine the velocity of sound through air as an ideal gas, evaluate the parameters in (9) via the ideal gas law.

$$\gamma = \frac{C_P}{C_V} = \frac{5/2\,R}{3/2\,R} = \frac{5}{3} \tag{10}$$

$$\kappa_T = \frac{-1}{V}\left(\frac{\partial V}{\partial P}\right)_T = \frac{-1}{V}\left(\frac{\partial\left(\frac{RT}{P}\right)}{\partial P}\right)_T = \frac{RT}{VP^2} = \frac{1}{P} \tag{11}$$

Plugging the results of (10) and (11) into (9) shows,

$$v_c^{air} = \sqrt{5/3\,PV} = \sqrt{5/3\,RT} \tag{12}$$

Adding the molecular weight of air to (12) to give velocity in m/s and evaluating,

$$v_c^{air} = \sqrt{\frac{\left(5/3\right)\left(8.314\text{ J/(mol K)}\right)\left(275.15\text{ K}\right)}{0.29\text{ kg/mol}}} = 363\text{ m/s} \tag{13}$$

Water

For water we use the steam table data to evaluate the heat capacity ratio, the specific volume at 2°C and 1 bar (0.1 MPa) and the isothermal compressibility.

$$\gamma = \frac{C_p^{water\ liquid}}{C_v^{water\ liquid}} = \frac{4.2140 \text{ kJ/kg K}}{4.2131 \text{ kJ/kg K}} = 1.0002 \tag{14}$$

The steam tables that you were given as a handout do not have the exact pressure and temperature points that you need to evaluate the velocity of sound through water. The zero pressure values in the steam table are italicized because they do not correspond to a real physical system since it is impossible to have water at 0 MPa. These values represent a state that has been extrapolated from physical data for states with P>0 MPa; however, the 0 MPa data can be used to estimate the 0.1 MPa data by interpolation between 2.5 and 0 MPa. A three step interpolation process is required to find the specific volume at 2°C and 0.1 MPa.

1) Interpolate between 0 and 20°C at 0 MPa to find,
V=1.0004 E-3 m^3/kg at 2°C and 0 MPa

2) Interpolate between 0 and 20°C at 2.5 MPa to find,
V=0.9992 E-3 m^3/kg at 2°C and 2.5 MPa

3) Interpolate between 0 and 20°C at 2.5 MPa to find,
V=1.0003 E-3 m^3/kg at 2°C and 0.1 MPa

The isothermal compressibility can be estimated using the results from the interpolated steam table data as follows,

$$\kappa_T = \frac{-1}{V}\left(\frac{\partial V}{\partial P}\right)_T \approx \frac{-1}{V}\left(\frac{\Delta V}{\Delta P}\right)_T = \frac{-1}{1.0003 \text{ E-3 m}^3\text{/kg}}\left(\frac{(0.9992 - 1.0004)\text{E-3 m}^3\text{/kg}}{2.5 \text{ E6 Pa}}\right)$$

$$\kappa_T = 4.8 \text{ E-10 Pa}^{-1}$$

$$v_c^{water} = \sqrt{\frac{\left(1.0003 \text{ E-3 m}^3\text{/kg}\right)\left(1.0002\right)}{4.8 \text{ E-10 Pa}^{-1}}} = 1444 \text{ m/s} \tag{15}$$

Calculate the distance between mother and pup.

The sound travels twice the distance, d, between mother and pup. The 3 second time delay that the mother seal hears represents the difference between the time it takes for the sound to travel twice the distance in air minus the time it takes for the sound to travel twice the distance in water. Thus,

$$2d = v_c^{water} t^{water} = v_c^{air} t^{air}$$

$$\Delta t = t^{air} - t^{water} = 3 \text{ s}$$

$$\Delta t = \frac{2d}{v_c^{air}} - \frac{2d}{v_c^{water}}$$

$$d = \frac{(\Delta t)\left(v_c^{water}\right)\left(v_c^{air}\right)}{2(v_c^{water} - v_c^{air})}$$

$$\boxed{d = 726 \text{ m} \approx 1/2 \text{ mile}}$$