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12.002 Physics and Chemistry of the Earth and Terrestrial Planets
Fall 2008

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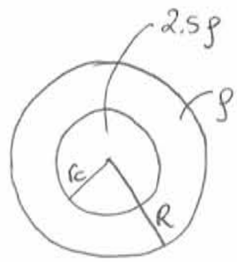
12.002 fall 2008 P-set 2 Solutions

$\boxed{\text{Problem 1}}$ 15p



Moment of inertia of a sphere with radius R, mass m and density ρ:

$$I = \frac{2}{5} m R^2 = \frac{2}{5} \rho \underbrace{\frac{4}{3}\pi R^3}_{\text{volume}} R^2 = \frac{8}{15}\pi\rho R^5$$

In our case the total moment of inertia of the planet is a sum of the moments of two spheres:

$$I_{total} = \underbrace{\frac{8}{15}\pi\rho R^5}_{\substack{\text{sphere of radius } R \\ \text{density } \rho}} + \underbrace{\frac{8}{15}\pi\,(2.5-1)\rho r_c^5}_{\substack{\text{sphere of radius } r_c \\ \text{density } (2.5-1)\rho}} = \frac{8}{15}\pi\rho\left(R^5 + \frac{3}{2} r_c^5\right)$$

By definition $L = \dfrac{I}{mR^2}$ or in our case $L = \dfrac{I_{total}}{M_{total} R^2}$

$$M_{total} = \frac{4}{3}\pi\rho R^3 + \frac{4}{3}\pi(2.5-1)\rho r_c^3 = \frac{4}{3}\pi\rho\left(R^3 + \frac{3}{2} r_c^3\right)$$

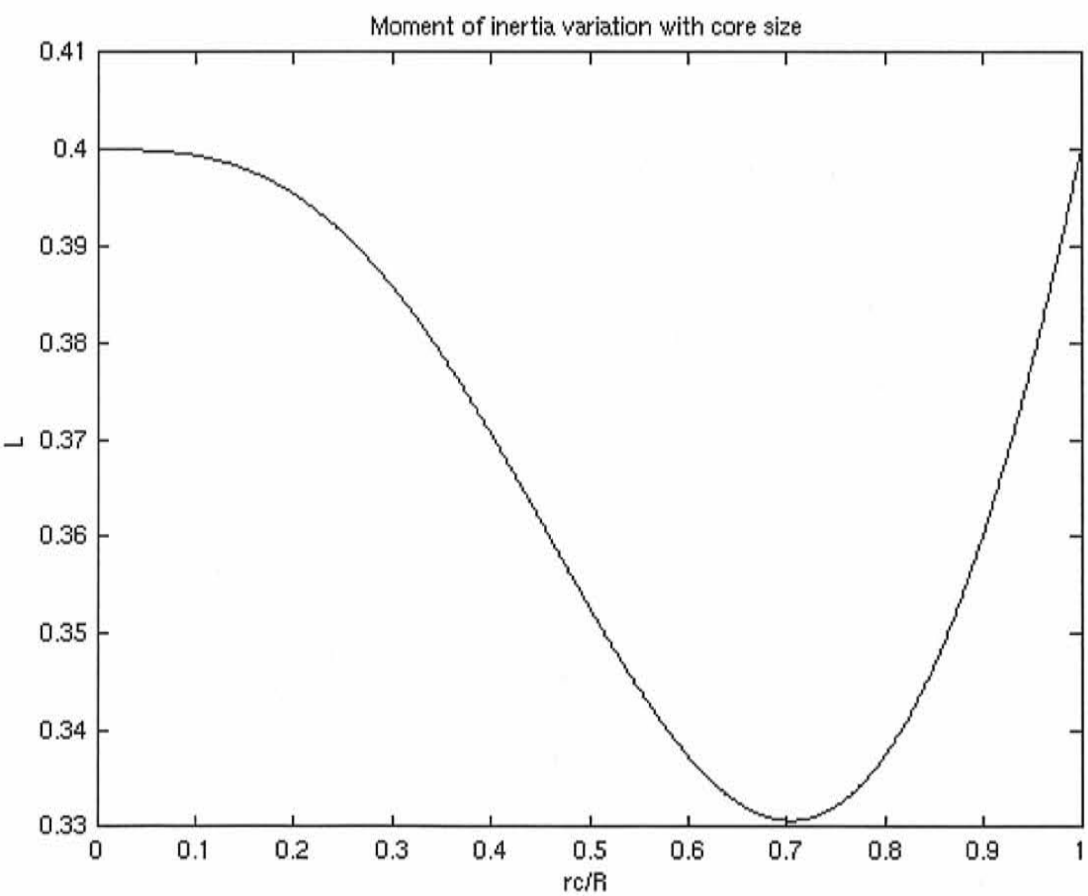
$$L = \frac{\frac{8}{15}\pi\rho\left(R^5 + \frac{3}{2} r_c^5\right)}{\frac{4}{3}\pi\rho\left(R^3 + \frac{3}{2} r_c^3\right)R^2} = \frac{2}{5}\left(\frac{R^5 + \frac{3}{2} r_c^5}{R^5 + \frac{3}{2} r_c^3 R^2}\right) = \frac{2}{5}\left(\frac{1 + \frac{3}{2}\frac{r_c^5}{R^5}}{1 + \frac{3}{2}\frac{r_c^3}{R^3}}\right)$$

$$= \frac{2}{5}\left(\frac{1 + 1.5x^5}{1 + 1.5x^3}\right) \quad \text{where } x = \frac{r_c}{R}$$

Plot $L(x)$ in MATLAB to get $L_{min} = 0.3307$ at $\boxed{x = 0.7040}$

```
% problem 1

rc_R=0:0.001:1;                                  % we'll plot for the radii ratio
                                                     form 0 to 1
L=(8/15 + 4/5.*rc_R.^5)./(4/3 + 2.*rc_R.^3);     % derived formula for L as a
                                                     function of rc/R

[miminum_value,index] = min(L)                   % finding the minimum value of
                                                     vector L
rc_R(index)                                      % the value of rc/R for which L is
                                                     minimum

plot(rc_R,L)                                     % plot the function
title('Moment of inertia variation with core size');
xlabel('rc/R');
ylabel('L');
```

```
miminum_value =

    0.3307

index =

   705

ans =

    0.7040
```



Problem 2 | 20p

a) using the plot you obtained (in MATLAB) read the values

Mars:	r_c/R	$r_c = r_c/R \cdot 3396$ km
	0.368	1249.7 km
	0.945	3209.2 km
Mercury:	r_c/R	$r_c = r_c/R \cdot 2440$ km
	0.704	1717.8 km * only one value here
Moon:	r_c/R	$r_c = r_c/R \cdot 1738$ km
	0.220	382.4 km
	0.987	1715.4 km

b) use the facts:

$M = \rho_{avg} \cdot \frac{4}{3}\pi R^3$ where M - total mass of a planet
ρ_{avg} - average density } given
R - total radius

$I = L \cdot M \cdot R^2$

to find total mass M and total moment of inertia I of each planet

then use the equations from the notes:

$$\begin{cases} I = \frac{8}{15}\pi\left(\rho_m R^5 + \rho_c r_c^5 - \rho_m r_c^5\right) \\ M = \frac{4}{3}\pi\left(\rho_m R^3 + \rho_c r_c^3 - \rho_m r_c^3\right) \end{cases}$$

ρ_m - mantle density
ρ_c - core density
r_c - core radius

and notice that you have given: I, M, R and r_c
and the only unknowns are ρ_m and ~~ρ_m~~ ρ_c.
this system of equations is therefore solvable.

From the equation for mass we can get the expression for ρ_c:

$$\frac{3M}{4\pi} = \rho_m(R^3 - r_c^3) + \rho_c r_c^3$$

$$\boxed{\rho_c = \frac{\frac{3M}{4\pi} - \rho_m\left(R^3 - r_c^3\right)}{r_c^3}}$$

To get the expression for ρ_m we manipulate the I - equation:

$$I = \frac{8}{15}\pi\left(\rho_c r_c^5 + \rho_m\left(R^5 - r_c^5\right)\right)$$

↑ plug in the derived ρ_c

$$\frac{15I}{8\pi} = \frac{\frac{3M}{4\pi} - \rho_m\left(R^3 - r_c^3\right)}{r_c^3} r_c^5 + \rho_m\left(R^5 - r_c^5\right)$$

$$\frac{15I}{8\pi} = \frac{3M}{4\pi} r_c^2 - \rho_m\left(R^3 - r_c^3\right)r_c^2 + \rho_m\left(R^5 - r_c^5\right)$$

$$\frac{15I}{8\pi} - \frac{3Mr_c^2}{4\pi} = \rho_m\left(-R^3 r_c^2 + \cancel{r_c^5} + R^5 - \cancel{r_c^5}\right)$$

$$\frac{15I - 6Mr_c^2}{8\pi} = \rho_m\left(R^5 - R^3 r_c^2\right)$$

$$\boxed{\rho_m = \frac{15I - 6Mr_c^2}{8\pi\left(R^5 - R^3 r_c^2\right)}}$$

For each planet plug in appropriate values first into the equation for ρ_m and then into ρ_c and then choose the results closest to $\rho_m = 3400\ kg/m^3$ and $\rho_c = 8000\ kg/m^3$.

See ~~4~~ three next pages for the solutions for each planet - correct answers are shaded in grey.

```
% Mars

rc_R = [0.368 0.945];        % ratio of core radius to total radius as read from the
                               graph to problem 1
R = 3396;                    % total radius as given in the problem [km]
ro_avg = 3933;               % average density as given in the problem [kg/m^3]
L = 0.376;                   % L given as given in the problem

rc = rc_R * R                % core radius [km]

M = ro_avg*4/3*pi*R^3*10^9; % total mass [kg]

I = L*M*R^2;                 % moment of inertia

% mantle density
rom = (15*I - 6*M.*rc.^2)./(8*pi*(R^5 - R^3.*rc.^2));   % [kg/km^3]
ro_m = 10^-9.*rom                                        % [kg/m^3]

% core density
ro_c = 10^-9.*(3*M/(4*pi) - rom.*(R^3 - rc.^3))./rc.^3  % [kg/m^3]
```

```
rc =

   1.0e+03 *

    1.2497    3.2092
```

$r_c =$ 1249.7 km

```
ro_m =

   1.0e+03 *

    3.6601    1.7271
```

$\rho_m =$ 3660.1 kg/m^3

```
ro_c =

   1.0e+03 *

    9.1369    4.3410
```

$\rho_c =$ 9136.9 kg/m^3

```
% Mercury

rc_R = 0.704;                % ratio of core radius to total radius as read from the
                               graph to problem 1
R = 2440;                    % total radius as given in the problem [km]
ro_avg = 5427;               % average density as given in the problem [kg/m^3]
L = 0.33;                    % L given as given in the problem

rc = rc_R * R                % core radius [km]

M = ro_avg*4/3*pi*R^3*10^9; % total mass [kg]

I = L*M*R^2;                 % moment of inertia

% mantle density
rom = (15*I - 6*M.*rc.^2)./(8*pi*(R^5 - R^3.*rc.^2));   % [kg/km^3]
ro_m = 10^-9.*rom                                        % [kg/m^3]

% core density
ro_c = 10^-9.*(3*M/(4*pi) - rom.*(R^3 - rc.^3))./rc.^3  % [kg/m^3]
```

```
rc =

   1.7178e+03

ro_m =

   3.5441e+03

ro_c =

   8.9406e+03
```

$r_c =$ 1717.8 km

$\rho_m =$ 3544.1 kg/m3

$\rho_c =$ 8940.6 kg/m3

```
% Moon

rc_R = [0.220 0.987];         % ratio of core radius to total radius as read from the
                                graph to problem 1
R = 1738;                     % total radius as given in the problem [km]
ro_avg = 3350;                % average density as given in the problem [kg/m^3]
L = 0.394;                    % L given as given in the problem

rc = rc_R * R                 % core radius [km]

M = ro_avg*4/3*pi*R^3*10^9; % total mass [kg]

I = L*M*R^2;                  % moment of inertia

% mantle density
rom = (15*I - 6*M.*rc.^2)./(8*pi*(R^5 - R^3.*rc.^2));   % [kg/km^3]
ro_m = 10^-9.*rom                                         % [kg/m^3]

% core density
ro_c = 10^-9.*(3*M/(4*pi) - rom.*(R^3 - rc.^3))./rc.^3  % [kg/m^3]
```

```
rc =

   1.0e+03 *

   0.3824    1.7154
```

$r_c =$ 382.4 km

```
ro_m =

   1.0e+03 *

   3.2972    1.4047
```

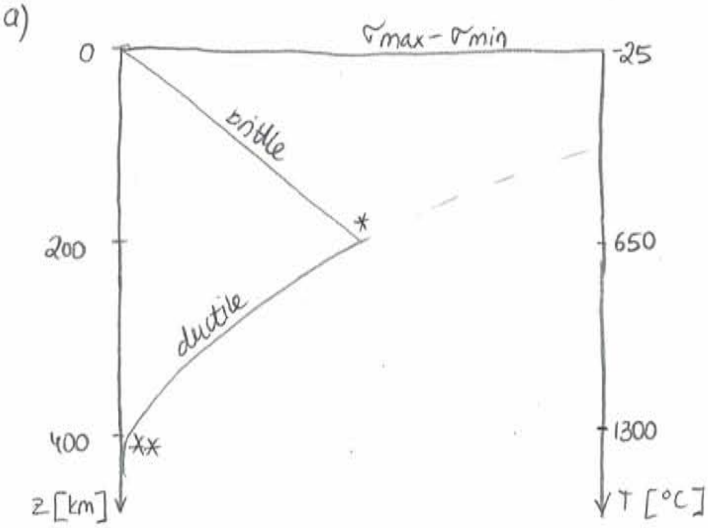
$\rho_m =$ 3297.2 kg/m³

```
ro_c =

   1.0e+03 *

   8.2564    3.4279
```

$\rho_c =$ 8256.4 kg/m³

Problem 3 10 p

a)



* change from brittle to ductile regime at 200 km or 650 °C

** deformation becomes negligible at the base of litosphere at 1300°C or $\boxed{392\ \text{km}}$

determined assuming linear thermal gradient

$$\frac{\partial T}{\partial z} = \frac{675\,K}{200\ km} = 3.375\ K/km$$

$$z = \frac{1325\ °C}{3.375\ K/km} = 392.6\ km$$

b) $K = 5\ \frac{W}{mK}$

$\frac{\partial T}{\partial z} = 3.375\ \frac{K}{km}$

Using the formula for heat flow from the notes we get

$$Q_{moon} = K \cdot \frac{\partial T}{\partial z} = 5\ \frac{W}{m\cancel{K}} \cdot 3.375 \cdot 10^{-3}\ \frac{\cancel{K}}{m} = \boxed{0.0169\ \frac{W}{m^2}}$$

Using the approximation for $\frac{\partial T}{\partial z}$ of the Earth we used in class:

$$Q_{earth} = 5\ \frac{W}{mK} \cdot 10\ \frac{K}{km} = \boxed{0.05\ \frac{W}{m^2}}$$

Q_{moon} is about 3 times smaller than Q_{earth}

This could indicate much thicker lithosphere on the Moon.

Problem 4 | 10p

Heat produced by a person during 1 day:

$Q = 2 \cdot 10^6$ cal $= 2 \cdot 10^6 \cdot 4.18\,J = 8.36 \cdot 10^6 J$

Time in seconds.

1 day $= 24 \cdot 60 \cdot 60 s = 86400 s = t$

Surface through which this heat flows:

$S = 2 m^2$

heat flow $\quad Q_{human} = \dfrac{Q}{tS} = \dfrac{8.36 \cdot 10^6 J}{8.64 \cdot 10^4 s \cdot 2m^2} = \boxed{48.4 \ \dfrac{W}{m^2}}$

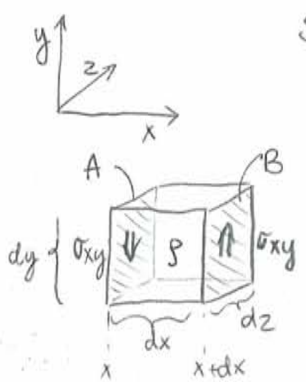
From previous problem we know $\quad Q_{earth} = 0.05 \ \dfrac{W}{m^2}$

area on the surface of Earth with ~~the~~ Q_{human}

$$A = \frac{Q_{human}}{Q_{earth}} \cdot S = \frac{48.4}{0.05} \cdot 2m^2 = \boxed{1936 m^2} = 44m \times 44m$$

Typical soccer field is around $7600 m^2$, but I guess you could still play on $2000 m^2$...

Problem 5 | 15p



S-wave vibrates in y-direction
propagates in x-direction

* remember stress $= \dfrac{\text{force}}{\text{area}}$

Start with $F = ma$

$$F = \underbrace{\sigma_{xy}(x+dx)\,dy\,dz}_{\text{force acting on side B}} - \underbrace{\sigma_{xy}(x)\,dy\,dz}_{\text{force acting on side A}} \ ^*$$

$m = \rho \, dx\,dy\,dz$

$a = \dfrac{\partial^2 v}{\partial t^2} \quad$ - as S-wave vibrates in y-direction

Equating F=ma gives us:

$$\sigma_{xy}(x+dx)\,\cancel{dydz} - \sigma_{xy}(x)\,\cancel{dydz} = \rho\, dx\cancel{dydz} \cdot \frac{\partial^2 v}{\partial t^2}$$

** $$\frac{\sigma_{xy}(x+dx)-\sigma_{xy}(x)}{dx} = \rho \frac{\partial^2 v}{\partial t^2}$$

** note the definition of a derivative

$$\frac{\partial \sigma_{xy}}{\partial x} = \rho \frac{\partial^2 v}{\partial t^2}$$

From the notes we have: $\sigma_{xy} = 2\mu\varepsilon_{xy}$

and $\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$= \mu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x \partial y}}}_{***}^{\nearrow 0} + \frac{\partial^2 v}{\partial x^2}\right)$$

*** S-wave does not vibrate in x-direction $u=0$

$$\boxed{\frac{\partial^2 v}{\partial t^2} = \frac{\mu}{\rho}\frac{\partial^2 v}{\partial x^2}}$$

We can express the displacement of a wave packet propagating in positive x direction as:

$v(x - v_s t)$

Then $$\frac{\partial^2 v}{\partial x^2} \overset{*}{=} \frac{\partial}{\partial x}\left(v' \cancel{\frac{\partial x}{\partial x}}^{\nearrow 1}\right) = \frac{\partial}{\partial x} v' = v'' \frac{dx}{\partial x} = v''$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial t}\left(v' \frac{\partial(-v_s t)}{\partial t}\right) = \frac{\partial}{\partial t}\left(v'(-v_s)\right) = v'' v_s^2$$

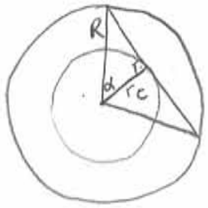
* using chain rule

Plug this into the derived equation:

$$\cancel{v''} v_s^2 = \frac{\mu}{\rho}\cancel{v''}$$

$$\boxed{v_s = \sqrt{\frac{\mu}{\rho}}}$$

Problem 6   15 p



$\alpha = \frac{103^\circ}{2} = 51.5^\circ$ $R = 6371$ km

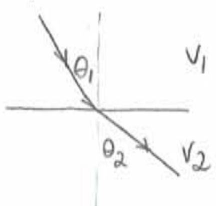
Using basic trigonometry:

$\cos\alpha = \frac{r_c}{R} \Rightarrow r_c = R\cos\alpha$

$\boxed{r_c = 3966 \text{ km}}$

The real core radius of Earth is about 3400 km, so our estimate is much too large.

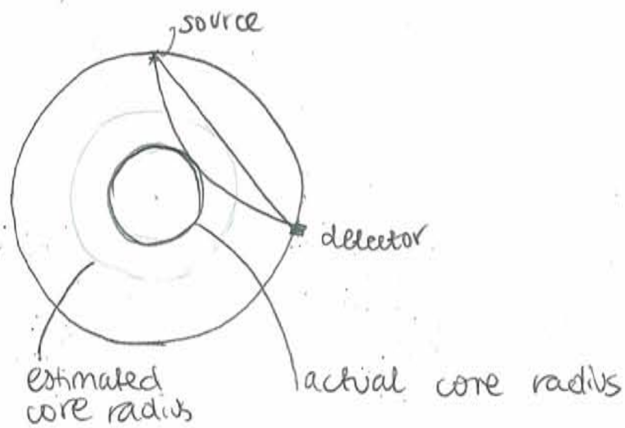
It is due to the fact that we have assumed uniform v_p (P-wave velocity) through mantle, while in fact v_p is increasing with depth.
According to Snell's law ray gets refracted when it passes throug a boundary of different v_p. In our case:



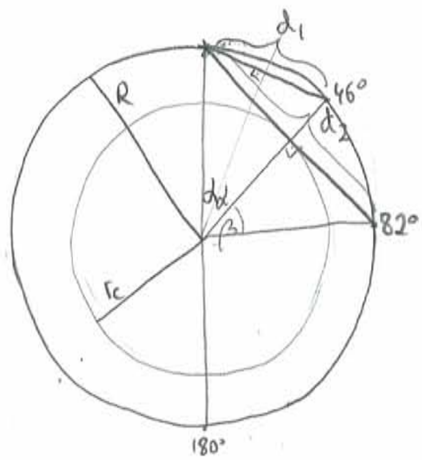
$v_2 > v_1 \Rightarrow \theta_2 > \theta_1$

as $\frac{v_1}{\sin\theta_1} = \frac{v_2}{\sin\theta_2}$

This causes that the rays are not linear, but concave-up
If the rays are not straight lines, but curves, then a ray detected at the critical angle points to the fact that the core/mantle boundary is <u>deeper</u>



Problem 7 15 p



Paths of the ray through mantle:

$$\sin\alpha = \frac{d_1/2}{R} \Rightarrow d_1 = 2R\sin\alpha$$

$$\alpha = \frac{46}{2} \quad R = 2000\,km$$

$$d_1 = 1562.9\ km$$

$$\sin\beta = \frac{d_2/2}{R} \Rightarrow d_2 = 2R\sin\beta$$

$$\beta = \frac{82}{2}$$

$$d_2 = \cancel{26}\ 2624.2\ km$$

mantle P-wave velocity

$$v_p = \frac{d_1}{t_1} = \frac{1562.9\ km}{312\ s} = 5.0094\ \frac{km}{s}$$

$$v_p = \frac{d_2}{t_2} = \frac{2624.2\ km}{525\ s} = 4.9985\ \frac{km}{s}$$

$$\boxed{V_p = 5.00\ \frac{km}{s}}$$

core radius (analogically to previous problem, with critical angle 83°)

$$r_c = R\cos\frac{83}{2}$$

$$\boxed{r_c = 1497.9\ km}$$

the core P-wave velocity:

The PKP ray that passes throug the planet center and is detected on the antipodes of the epicentre travels through $2r_c$ of core and with speed v_c $2R-2r_c$ of mantle with speed $V_m = 5.00\ \frac{km}{s}$ and it takes $t = 300s$

$$\frac{2r_c}{v_c} + \frac{2R-2r_c}{V_m} = t \qquad \frac{V_m t - 2R + 2r_c}{V_m} = \frac{2r_c}{V_c}$$

$$v_c = \frac{2r_c V_m}{V_m t - 2R + 2r_c} = \frac{2995.8\,km \cdot 5.00\ \frac{km}{s}}{5.00 \cdot \frac{km}{s}\ 300s - 1004.2\,km} = \boxed{30.2\ \frac{km}{s}}$$

This planet must have a <u>solid</u> core, as only in solid this crazy fast P-wave propagation speed of over 30 $\frac{km}{s}$ would be possible.