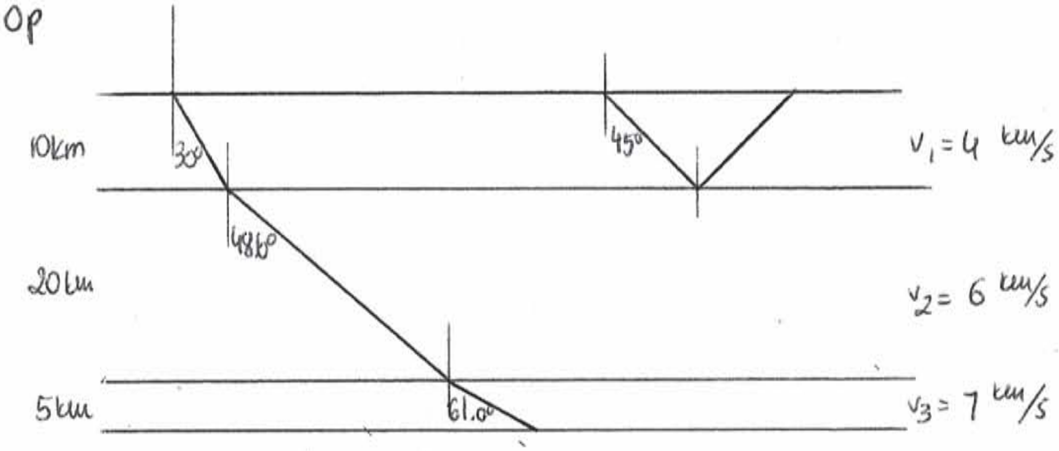
12.002 Physics and Chemistry of the Earth and Terrestrial Planets
Fall 2008

12.002 fall 2008 P-set 3 Solutions

① 10p



Snell's law:

$$\frac{v_1}{\sin\alpha_1} = \frac{v_2}{\sin\alpha_2} \qquad \alpha_2 = \arcsin\left(\frac{v_2 \sin\alpha_1}{v_1}\right)$$

For $\alpha_1 = 30°$, $\alpha_2 = \arcsin \frac{6\sin 30}{4} \approx 48.6°$

$\alpha_3 = \arcsin \frac{7 \sin\alpha_2}{6} \approx 61.0°$

For $\alpha_1 = 45°$, $\alpha_2 = \arcsin \frac{6 \sin 45}{4} \Rightarrow$ is not real
no refraction - just reflection

② 15p

P-wave:

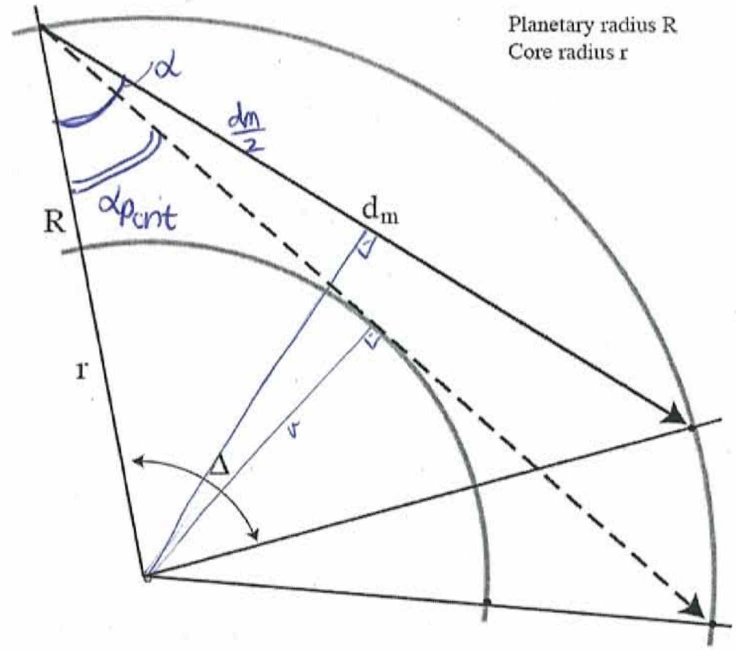
- the curve is concave-down
  → it's a result of the fact that P-wave velocity increases for greater depth but also of the geometry, as distance travelled by a ray is not a linear function of the angular distance (note that the plots you got in problem 3 are also curves, even though you assumed constant P-wave velocity in the mantle).
- don't go beyond ~103°
  → core shadow, 103° is a result of core to total radius ratio

PKP-wave:

- gap in Δ between P & PKP-waves
  → result of refraction at the core ~~and due to~~ towards the normal due to the fact that $V_m > V_c$ for P-wave

③ We'll find expressions for Δ and t for P-and PKP-waves
35p as function of angle α – angle between ray leaving source and vertical.

P-wave



given: R, r, v_m, v_c

$\alpha_{Pcritical}$*: $\sin\alpha_{Pcrit} = \frac{r_c}{R}$ (trigonometry)

$\alpha_{Pcrit} = \arcsin\frac{r_c}{R}$

Δ : $\Delta = 180 - 2\alpha$ (Δ angles)

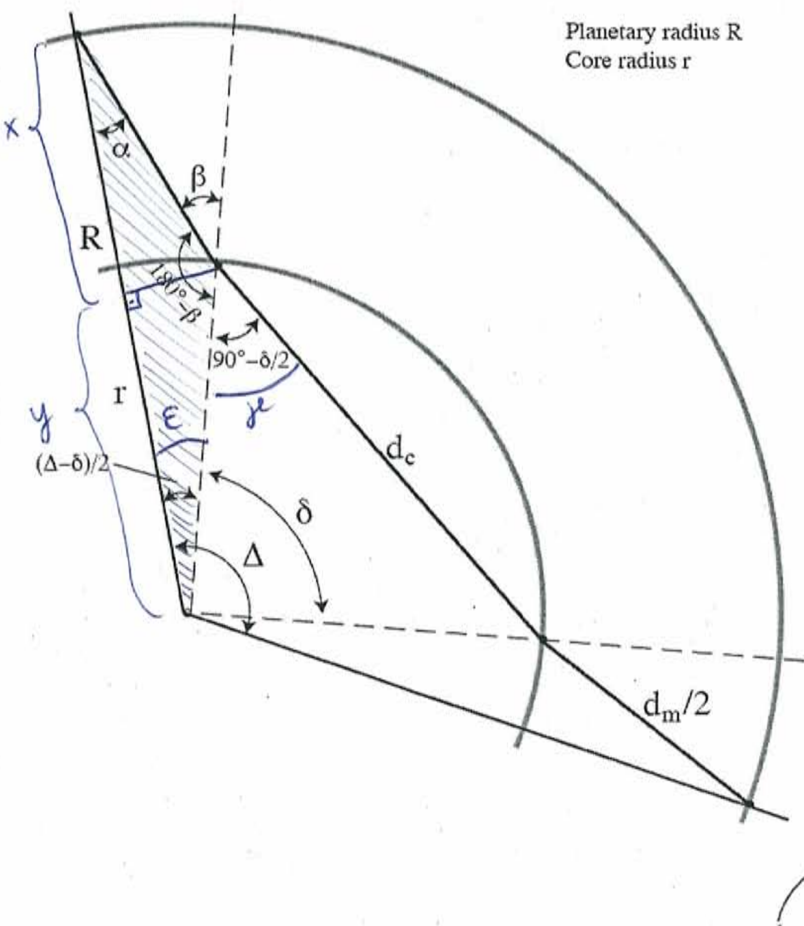
d_m : $\cos\alpha = \frac{d_m/2}{R}$

$d_m = 2R\cos\alpha$ (trig.)

t : $t = \frac{d_m}{v_m}$ (velocity eqn.)

Plot Δ vs t for $\alpha(\alpha_{Pcrit}, 90)$

PKP-wave



in our quest to ultimately express Δ and t as functions of α we'll find expressions for the following: $\varepsilon, d_m, d_c, \beta, \gamma$

By the law of sines for the shaded triangle we get:

$$\frac{\sin\varepsilon}{d_m/2} = \frac{\sin\alpha}{r} \Rightarrow \boxed{d_m = \frac{2r\sin\varepsilon}{\sin\alpha}}$$ 2

If a height ~~dpe~~ dropped on side R of the shaded triangle splits it into x and y, we get:

$$\begin{cases} x + y = R \quad \text{and} \\ x = \frac{d_m}{2}\cdot\cos\alpha \\ y = r\cos\varepsilon \end{cases}$$

$$\Rightarrow \frac{d_m}{2}\cos\alpha + r\cos\varepsilon = R$$

plug d_m we found here

$$\frac{r\sin\varepsilon}{\sin\alpha}\cos\alpha + r\cos\varepsilon = R$$

$$\frac{r\sin\varepsilon}{\sin\alpha}\cos\alpha + r\cos\varepsilon = R \quad /:r$$

$$\frac{\sin\varepsilon}{\sin\alpha}\cos\alpha + \cos\varepsilon = \frac{R}{r} \quad /\cdot\sin\alpha$$

$$\underbrace{\sin\varepsilon\cos\alpha + \cos\varepsilon\sin\alpha}_{\sin(\alpha+\varepsilon)} = \frac{R}{r}\sin\alpha$$

$$\sin(\alpha+\varepsilon) = \frac{R}{r}\sin\alpha$$

$$\boxed{\varepsilon = \arcsin\left(\frac{R}{r}\sin\alpha\right) - \alpha} \quad \boxed{1}$$

and here we have ε expressed as a function of α
plug it back to the equation for d_m to get $d_m(\alpha)$

β: $\quad 180 - \beta = 180 - \alpha - \varepsilon \qquad$ (triangle angles)

$$\boxed{\beta = \alpha + \varepsilon} \quad \boxed{3}$$

γ: $\quad \frac{v_m}{\sin\beta} = \frac{v_c}{\sin\gamma} \qquad$ (Snell's law)

$$\boxed{\gamma = \arcsin\left(\frac{v_c\sin\beta}{v_m}\right)} \quad \boxed{4}$$

d_c: $\quad \cos\gamma = \frac{d_c/2}{r_c} \qquad$ (trig.)

$$\boxed{d_c = 2r_c\cos\gamma} \quad \boxed{5}$$

Δ: $\quad \Delta = 3\cdot 180 - 2\alpha_4 - 2\gamma - 2(180-\beta) \qquad$ (angles of pentagon)

$$\boxed{\Delta = 180 - 2\alpha + 2\beta - 2\gamma}$$

t: $\quad \boxed{t = \frac{d_m}{v_m} + \frac{d_c}{r_c}} \qquad$ (velocity eqn)

If we do the calculations in MATLAB in the order indicated in the square above each equation, then every value is defined.

For Mercury ($v_c < v_m$) plot Δ vs. t for $\alpha\,(0, \alpha_{\text{crit}})$

For Mars ($v_c > v_m$) we have to determine $\alpha_{\text{Kcritical}}$, as not for every incident angle of ray hitting the core-mantle boundary refraction will occur.

The condition for a ray hitting CMB and refracting into the core is that $\gamma < 90°$.

$$\frac{v_m}{\sin\beta} = \frac{v_c}{\sin 90°} \qquad \text{(Snell's law)}$$

$$\sin\beta = \frac{v_m}{v_c}$$

$$\boxed{\alpha_2 = \arcsin\frac{v_m}{v_c}}$$

Now, using equations $\boxed{1}$ and $\boxed{3}$ calculate critical angle $\alpha_{PKPcrit}$ and plot Δ vs. t for $\alpha\,(0, \alpha_{PKPcrit})$ for Mars.

From the plots (see attached) we can see that Mercury is similar to the Earth: there is a gap between P and PKP arrivals as a result of $v_m > v_c$. This also hints that the core is liquid.
Conversely, for Mars we see an overlap of P & PKP waves and combined with the fact that $v_c > v_m$ points to the fact that Mars' interior is probably solid.

```
% Mercury (green)

vm = 8;
vc = 6;
r = 1800;
R = 2440;

% P-wave
```

calculate α_p critical

```
alpha_P_crit = asind(r/R);
```

α — define α vector

```
alpha = alpha_P_crit:0.01:90;
```

Δ

```
delta = 180 - 2.*alpha;

dm = 2*R.*cosd(alpha);
```

t

```
t = dm/vm;

plot(delta,t,'-.');
title('P and PKP travel times as function of angular distance for Merury [dash-dot] and Mars [cont]');
xlabel('angular distence [degrees]');
ylabel('travel time [s]');
hold on

% PKP-wave
```

α

```
alpha = 0:0.1:alpha_P_crit;
```

1

```
epsilon = asind(R/r.*sind(alpha)) - alpha;
```

2

```
dm = (2*r.*sind(epsilon))./sind(alpha);
```

3

```
beta = epsilon + alpha;
```

4

```
gamma = asind((vc.*sind(beta))/vm);
```

5

```
dc = 2*r.*cosd(gamma);
```

Δ

```
delta = 180 + 2*beta - 2*alpha - 2*gamma;
```

t

```
t = dc/vc + dm/vm;

plot(delta,t,'-.');
hold on
```

```
% Mars (red)

vm = 10;
vc = 11;
r = 1200;
R = 3400;

% P-wave ___________________________________________

alpha_P_crit = asind(r/R);
```

calculate $\alpha_{P critical}$

```
alpha = alpha_P_crit:0.01:90;
```

define α vector

```
delta = 180 - 2.*alpha;

dm = 2*R.*cosd(alpha);

t = dm/vm;

plot(delta,t);
hold on

% PKP-wave ___________________________________________

alpha = 0:0.01:alpha_P_crit;

epsilon = asind(R/r.*sind(alpha)) - alpha;

dm = (2*r.*sind(epsilon))./sind(alpha);

beta = epsilon + alpha;

for n=1:length(beta)
    if (beta(n) <= asind(vm/vc))
        i = n;
    end
end
```

calculate new constraint for $\alpha \to \alpha_{PKP critical}$ and recalculate everything with the new vector

```
alpha_PKP_crit = alpha(i);

alpha = 0:0.1:alpha_PKP_crit;

epsilon = asind(R/r.*sind(alpha)) - alpha;

dm = (2*r.*sind(epsilon))./sind(alpha);

beta = epsilon + alpha;

gamma = asind((vc.*sind(beta))/vm);

dc = 2*r.*cosd(gamma);

delta = 180 + 2*beta - 2*alpha - 2*gamma;

t = dc/vc + dm/vm;

plot(delta,t);
```

(4) 10p

integrate over depth! not radius

$$P = \int \rho g \, dz \qquad g = \frac{M_{in} G}{r^2} = \frac{4}{3}\pi r^3 \rho \frac{G}{r^2} = \frac{4}{3}\pi G \rho r \qquad r = R - z$$

$$P = \int_0^z \rho \frac{4}{3}\pi G \rho r \, dz = \frac{4}{3}\pi G \rho^2 \int_0^z (R - z)\, dz = \frac{4}{3}\pi G \rho^2 \left[Rz - \frac{z^2}{2} \right]_0^z =$$

$$= \boxed{\frac{4}{3}\pi G \rho^2 \left(Rz - \frac{z^2}{2} \right)}$$

```
% Mars

R = 3400000;                     % [m]
rho = 3930;                      % [kg/m^3]
G = 6.67 * 10^-11;               % [m^3/(kg s)]

z = 0:1000:R;

P1 = 4/3*pi*rho^2*G.*(R*z -0.5.*z.^2);

P2 = P1 - 24*10^9;
P3 = abs(P2);
[a,b] = min(P3);
```

} find the pressure in Mars closest to 24GPa

```
Mars_depth_of_24GP = z(b)/1000
Mars_r_of_24GP = R/1000 - Mars_depth_of_24GP


% Moon

R = 1740000;                     % [m]
rho = 3350;                      % [kg/m^3]
G = 6.67 * 10^-11;               % [m^3/(kg s)]

z = 0:1000:R;

P1 = 4/3*pi*rho^2*G.*(R*z -0.5.*z.^2);

P2 = P1 - 24*10^9;
P3 = abs(P2);
[a,b] = min(P3);
```

} find the pressure in Moon closest to 24 GPa

```
Moon_max_pressure = P1(length(P1))

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

Mars_depth_of_24GP = 2739

Mars_r_of_24GP = 661

Moon_max_pressure = 4.7465e+09
```

} 24GPa exists in Mars at r = 661 km, which is within core. There is probably no perovskite on Mars, as core has different composition (probably mostly metallic) and would not support perovskite due to composition constraints.

Moon does not have enough pressure for perovskite transition → no perovskite.

⑤ 10p

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial z^2}$$

$$T(z,t) = T_S + (T_0 - T_S)\, erf\left(\frac{z}{2\sqrt{Kt}}\right) = T_S + (T_0 - T_S)\frac{2}{\sqrt{\pi}}\int_0^{\frac{z}{2\sqrt{Kt}}} e^{-y^2}\,dy$$

Note that : $\frac{d}{dx} erf(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$

$$LHS = \frac{\partial T}{\partial t} = \frac{\cancel{2}(T_0 - T_S)}{\sqrt{\pi}} \cdot \frac{z}{\cancel{2}\sqrt{K}}\left(-\frac{1}{2}t^{-\frac{3}{2}}\right) e^{-\frac{z^2}{4Kt}}$$

$$= \boxed{-\frac{(T_0 - T_S)\, z}{2\sqrt{\pi K t}\, t} e^{-\frac{z^2}{4Kt}}}$$

$$\frac{\partial T}{\partial z} = \frac{\cancel{2}(T_0 - T_S)}{\sqrt{\pi}} \frac{1}{\cancel{2}\sqrt{Kt}} e^{-\frac{z^2}{4Kt}}$$

$$RHS = K\frac{\partial^2 T}{\partial z^2} = \cancel{K}\frac{T_0 - T_S}{\sqrt{\pi K t}}\left(-\frac{\cancel{2}z}{2\cancel{4K}t}\right) e^{-\frac{z^2}{4Kt}}$$

$$= \boxed{-\frac{(T_0 - T_S)\, z}{2\sqrt{\pi K t}\, t} e^{-\frac{z^2}{4Kt}}}$$

LHS=RHS Q.E.D.

⑥ 15p

a) $z_b = 50\ km = 50 \cdot 10^3 m$
$K = 10^{-6}\ m^2/s$
$T_m = T_0 = 1500°C$

At the base of the lithosphere :

$$T(z_b, t) = T_S + (T_0 - T_S)\, erf\left(\frac{z_b}{2\sqrt{Kt}}\right)$$

We assume that temperature T_b at the base of the lithosphere is $0.99\, T_0$.

$$\frac{0.99 T_0 - T_S}{T_0 - T_S} = erf\left(\frac{z_b}{2\sqrt{Kt}}\right)$$

for our rough estimate we can ignore T_S in our calculation

$$0.99 = erf\left(\frac{z_b}{2\sqrt{Kt}}\right)$$

from the tables of erf we get the value that gives 0.99 → $\frac{z_b}{2\sqrt{Kt}} = 1.82$

in

$$t = \left(\frac{2b}{2 \cdot 1.82\sqrt{K}}\right)^2 = 1.8868 \cdot 10^{14}\, s \approx \boxed{5.98\ My}$$

According to this estimation Mercury's lithosphere is very young ≈ 6 million years.

b) If the lithosphere is in fact 4.0 Ga, it would mean that it should be much thicker if only conduction was at work.

We can estimate the thickness for 4.0 Ga old lithosphere in the same way

$$0.99 = \operatorname{erf}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\frac{z}{2\sqrt{\kappa t}} = 1.82 \quad \Rightarrow \quad z = 2 \cdot 1.82 \cdot \sqrt{\kappa t}$$

$$z \approx 1293\ km$$

If the lithosphere is not 1293 km as expected, but much thinner (50 km) there must have been some process that thinned it from below or prevented cooling: remelting of lower lithosphere, hot plumes in the mantle.