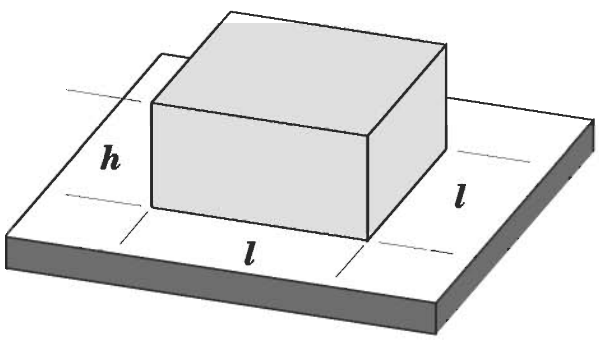
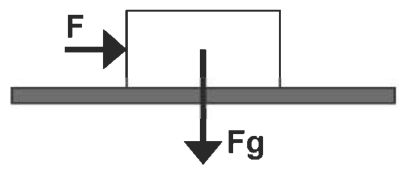
Problem 1

Turcotte & Schubert 2-8 (pg. 80), 2-10 in version 2

Consider a rectangular block of rock with height of 1 meter and horizontal dimensions of 2 meters. The density of the rock in 2.75 Mg/m^3. If the coefficient of friction is 0.8, what force is required to slide the rock over a horizontal surface?

Solution





The given quantities are, where g is gravitational acceleration:

$$\rho = 2.75\mathrm{Mg/m^3} = 2750\mathrm{kg/m^3}$$
$$h = 1.0\mathrm{m} \quad l = 2.0\mathrm{m}$$
$$\mu = 0.8$$
$$g = 9.80\mathrm{m/s^2}$$

We first calculate the shear traction required to move the block, given the normal traction due to gravity, by:

$$\tau = \mu\,\sigma_n = \mu\,\frac{\text{density} \times \text{volume} \times g}{\text{area}} = \mu\,\frac{\vec{F}_g}{A} = \mu\,\frac{\rho gllh}{ll}$$
$$\tau = 0.8\frac{2750 \cdot 9.81 \cdot 2 \cdot 2 \cdot 1}{2 \cdot 2}\mathrm{kg/s \cdot m^2} = 2.16 \times 10^4\mathrm{N/m^2}$$

And the force is found by multiplying the shear traction by the area it is acting over (the bottom):

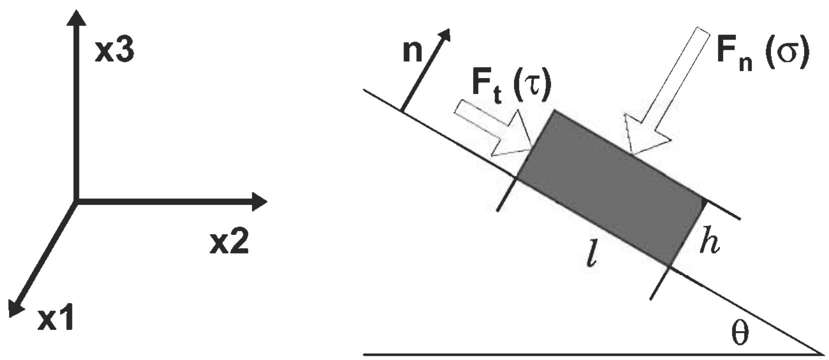
$$\vec{F} = \tau \cdot l \cdot l = (2.16 \times 10^4\mathrm{N/m^2}) \times (2 \cdot 2\mathrm{m^2}) = 8.63 \times 10^4\mathrm{N}$$

Problem 2

Consider a rock mass of density ρ and thickness h resting on an inclined plane, with the dip angle of the plane θ shown in the figure. The plane is just steep enough that frictional sliding continues after it begins.

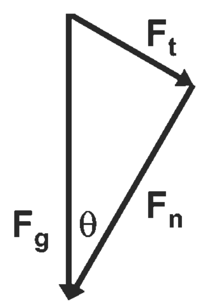
a) Calculate the relation between the coefficient of friction f and θ.

b) Give the components of the normal vector to the plane, $\hat{n}$, in terms of θ.



Solution

a) A free body diagram of the forces in this problem is:



First calculate the forces and tractions, where g is the acceleration due to gravity, and $V = h \cdot l \cdot w$ is volume (w is width in and out of the page):

$$\vec{F}_g = \rho V g = \rho h l w g$$

$$\sigma_n = \frac{\vec{F}_n}{lw} = \frac{\vec{F}_g \cos\theta}{lw} = \rho h g \cos\theta$$

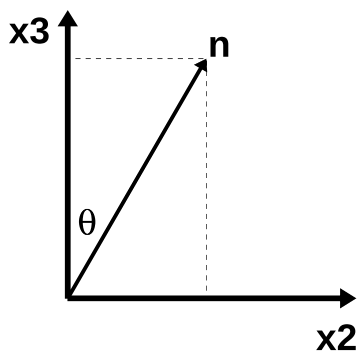
$$\tau = \frac{\vec{F}_t}{lw} = \frac{\vec{F}_g \sin\theta}{lw} = \rho h g \sin\theta$$

And since the block started sliding we know that

$$\tau = f\sigma_n \Rightarrow f = \frac{\tau}{\sigma_n} = \frac{\rho h g \sin\theta}{\rho h g \cos\theta} = \frac{\sin\theta}{\cos\theta}$$

$$f(\theta) = \tan\theta$$

b)



The normal vector to the plane, $\hat{n}$, in terms of θ, is

$$\hat{n} = (0)\,\hat{e}_1 + (\sin\theta)\,\hat{e}_2 + (\cos\theta)\,\hat{e}_3$$

or

$$\hat{n} = (0, \sin\theta, \cos\theta)$$

Note that it is required for a normal vector to be unit length.

Problem 3

It is a good approximation in many geodynamical situations that variations in topography are compensated isostatically. That is, above the depth of compensation, the weight of the material in any column is a constant. The purpose of this problem is to determine whether isostatic compensation and a state of lithostatic stress are compatible. As a specific example, we will consider the simplified model of a mid-oceanic ridge shown in the figure. Assume that the lithosphere has a uniform density, $\rho_l = 3300$ kg/m^3, which is slightly greater than that of the underlying asthenosphere, which has density $\rho_a = 3250$ kg/m^3. Assume that water has a density $\rho_w = 1000$ kg/m^3. The lithosphere has zero thickness under the ridge crest, and thickens as it cools to a constant thickness (say 135 km) far from the ridge. As a result of isostasy, the ridge is at an elevation which is higher that the ocean basin.

a) What is the elevation of the ridge if it is in isostatic equilibrium?

b) Assuming that the state of stress is lithostatic at both places, make a graph of the horizontal normal stress, σ_{xx}, as a function of depth beneath both the ridge crest (point R) and the abyssal plain (point P).

c) F_x, the horizontal force per unit length (into the page) acting on the lithosphere, can be determined by integrating σ_{xx} over the thickness of the lithosphere. For this problem, with constant densities, this integration is easy to do graphically. Consider a free body diagram of the lithosphere made by drawing a box with edges beneath points R and P. Determine the net horizontal force per unit length acting on the lithosphere if the assumption of lithostatic stress applies.

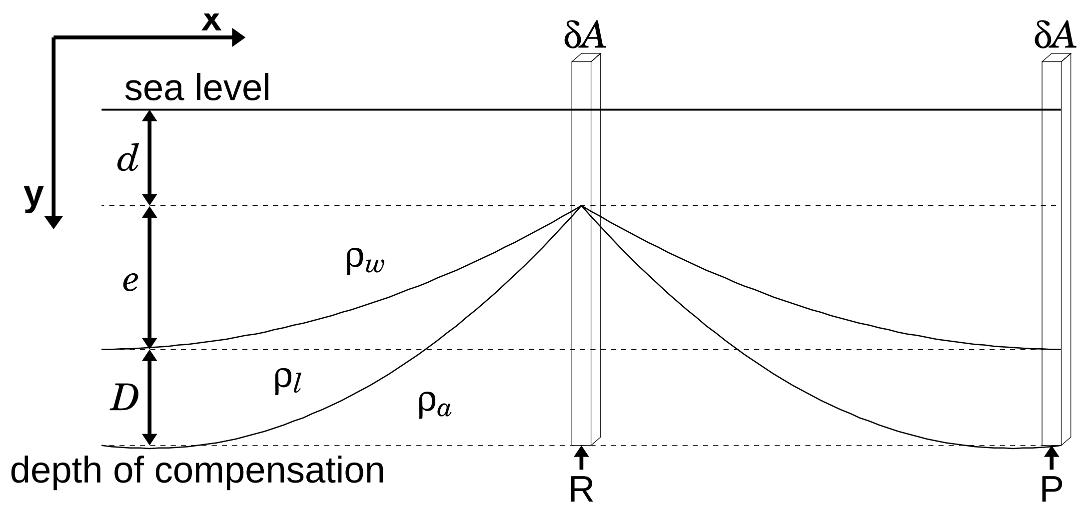
d) In order that there not be a net force acting on the lithosphere, the assumption of lithostatic stress must be modified. Calculate the magnitude of the average non-lithostatic stress, $\Delta\sigma_{xx}$, acting over the 135 km thickness of the lithosphere, required to balance the forces on the lithosphere.

e) How does the magnitude of $\Delta\sigma_{xx}$ compare to the average value of the lithostatic stress σ_{xx}?

f) If the departure from lithostatic stress, $\Delta\sigma_{xx}$, occurs in the old lithosphere, is it extensional of compressional? What if it occurs at the ridge?

Solution

Diagram not to scale.



$$_w = 1000\text{kg/m}^3, \quad \rho_l = 3300\text{kg/m}^3, \quad \rho_a = 3250\text{kg/m}^3$$

$$d = 2.5\text{km}, \quad D = 135\text{km}$$

a) For isostasy the weight of column R must equal the weight of column P above the depth of compensation, which is the depth below which the material in the two columns is the same.

$$w_R = d \cdot \delta A \cdot \rho_w \cdot g + (e + D) \cdot \delta A \cdot \rho_a \cdot g$$

$$w_P = (d + e) \cdot \delta A \cdot \rho_w \cdot g + D \cdot \delta A \cdot \rho_l \cdot g$$

where we assume that the cross sectional area of the columns, δA, is small, thus the lithosphere has thinned to zero in the column along the ridge. Since $w_P = w_R$ we can combine equations and solve for the unknown, e:

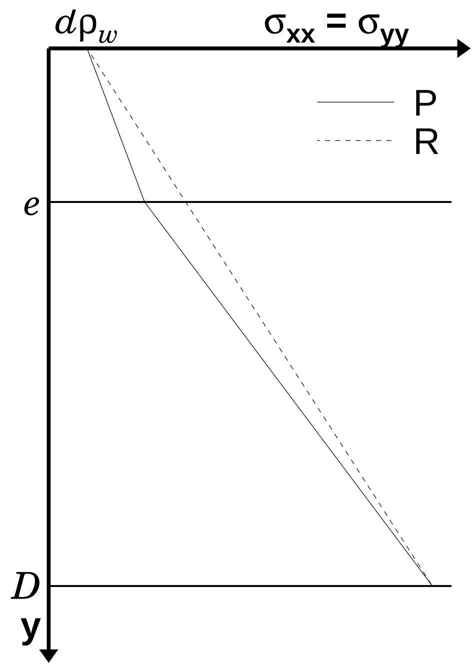
$$e = D \frac{_l - \rho_a}{\rho_a - \rho_w} = 135 \frac{3300 - 3250}{3250 - 1000} \text{km} = 3.0\text{km}$$

b) The equation for pressure, σ_{yy}, at depth y with pressure p_0 at $d = 0$ is $\sigma_{yy} = p_0 + \ \ g \cdot y$ (which is just an equation for a line), and since we have lithostatic stress we know that $\sigma_{xx} = \sigma_{yy}$. Setting depth d below sea level to $y = 0$, so that $p_0 = d\rho_w g$, we can construct equations for σ_{xx} under the ridge (point R) and the plain (point P):

$$\sigma^R_{xx}(y) = y\, \rho_a g + d\, \rho_w g, \ \ 0 \le y \le (e + D)$$

$$\sigma_{xx}^{P}(y) = \begin{cases} y\,\rho_w g + d\,\rho_w g & 0 \le y \le e \\ y\,\rho_l g + (d+e)\rho_w g - e\,\rho_l g & e \le y \le (e+D) \end{cases}$$

Where you need to find the intercept of the bottom equation from a two point line formula. A plot of these curves is (not to scale):



c) The net force per unit length will be the area between the two curves in the above graph. Note that this is force *per unit length* because when we integrate a stress ($\frac{\vec{F}}{A}$) over depth we result in a force per length since we essentially remove one dimension from the area. To solve this analytically we integrate the functions found above:

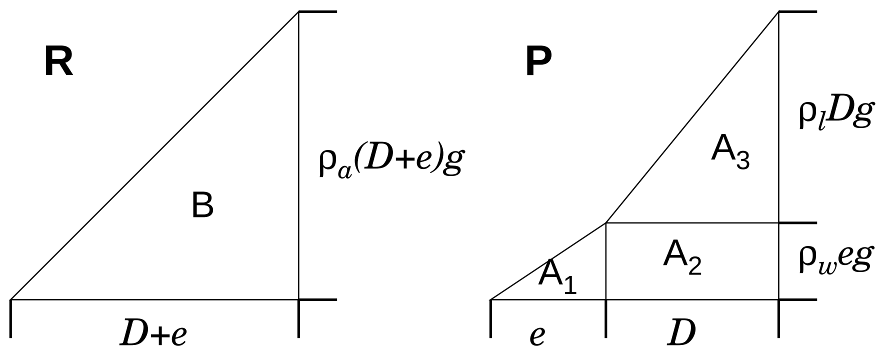
$$F_x = \frac{\vec{F}_x}{A} = \int_0^{D+e} \sigma_{xx}^{R} dy - \int_0^{D+e} \sigma_{xx}^{P}$$

$$F_x = \int_0^{D+e} [d\,\rho_w + y\,\rho_a] g\, dy - \int_0^{e} [d\,\rho_w + y\,\rho_w] g\, dy - \int_e^{D+e} [(d+e)\rho_w - e\,\rho_l + y\,\rho_l] g\, dy$$

This is relatively straight forward to solve, and after some calculus and algebra we get:

$$F_x = \frac{\rho_a}{2}(D+e)^2 g - \frac{\rho_l}{2} D^2 g - \frac{\rho_w}{2}(e^2 + 2eD) g$$

Before evaluating this notice that with these functions F_x can be also found by geometry — find the area under the R curve, and subtract the area under the P curve. Removing the rectangle common to both area, and rotating the axes 90° we can visualize the geometry as



where we can find the resultant force per unit area directly by taking the difference between the areas

$$F_x = \mathbf{B} - \mathbf{A_1} - \mathbf{A_2} - \mathbf{A_3}$$

$$\Rightarrow F_x = \frac{\rho_a}{2}(D+e)^2 g - \frac{\rho_l}{2} D^2 g - \frac{\rho_w}{2}(e^2 + 2eD)g$$

which is the same result as obtained with calculus (which is a good check on the work). Plugging in numbers and calculating we find

$$F_x = 4.6 \times 10^{12}\mathrm{N/m}$$

d) The magnitude of the average non-lithostatic stress is given as the force per unit width divided by the length that the force is acting over (ie. the force per unit area).

$$\Delta\sigma_{xx} = \frac{F_x}{D} = \frac{4.6 \times 10^{12}\mathrm{N/m}}{135 \times 10^3\mathrm{m}} = 3.41 \times 10^7\mathrm{Pa} = 34.1\mathrm{MPa}$$

e) Since stress increases lineraly with depth, the average value of the lithostatic stress is the average of the minimum and maximum values. The magnitude of σ_{yy} at the top of the oceanic lithosphere at point P is

$$\sigma_{yy}^{\min} = (d+e)\rho_w g = ([2.5+3.0] \times 10^3\mathrm{m})(1000\mathrm{kg/m^3})(9.81\mathrm{m/s^2}) = 54.0\mathrm{MPa}$$

The magnitude of σ_{yy} at the top of the oceanic lithosphere at point P is

$$\sigma_{yy}^{\max} = \sigma_{yy}^{min} + D\rho_l g = 5.4\times10^7\mathrm{Pa} + (135\times10^3\mathrm{m})(3300\mathrm{kg/m^3})(9.81\mathrm{m/s^2}) = 4.42\mathrm{GPa}$$

and since $\sigma_{yy} = \sigma_{xx}$ (for lithostatic stress) the average is

$$\overline{\sigma_{xx}} = \frac{\sigma_{yy}^{\max} - \sigma_{yy}^{\min}}{2} = 2.2\mathrm{GPa}.$$

Quantitatively

$$\left(\frac{\Delta\sigma_{xx}}{\overline{\sigma_{xx}}}\right) \times 100 = \left(\frac{34.1\text{MPa}}{2.2 \times 10^3\text{MPa}}\right) \times 100 = 1.6\%$$

Which is very small, and might be considered negligable.

f) All of the material from the ridge to the abyssal plain will tend to "slide" away from the ridge under its own weight, therefore, the non-lithostatic stress in the lithosphere at point P will be compressional. Alternatively, if the lithosphere is "sliding" away from the ridge, the non-lithostatic stress at point R will be tensional.