Problem 1

Write out symbolically the relationship between the traction vector acting on a surface, T_i, the normal vector describing the orientation of that surface, n_j, and the local stress tensor, σ_{ji}, using:

a) vector-matrix notation

b) the summation symbol Σ

c) Einstein summation notation

d) full explicit evaluation writing out each term

Solution

a) (NOTE: Since symbol notation is not convenient to write in TEX I will use $\vec{T}$, $\hat{n}$ and σ for a vector, unit length vector and tensor respectively.)

$$\vec{T} = \sigma^t \hat{n}$$

and since $\sigma^t = \sigma$, $\vec{T} = \sigma \hat{n}$.

b)

$$T_i = \sum_{j=1}^{3} \sigma_{ji} \hat{n}_j, \; i = 1, 2, 3$$

where I retained the hat on the unit length vector, this is optional but I find it to be convenient.

c)

$$T_i = \sigma_{ji} \hat{n}_j$$

where it is assumed that you sum over the dummy index j, and again it is assumed that $i = 1, 2, 3$.

d)

$$T_1 = \sigma_{11}\hat{n}_1 + \sigma_{21}\hat{n}_2 + \sigma_{31}\hat{n}_3$$

$$T_2 = \sigma_{12}\hat{n}_1 + \sigma_{22}\hat{n}_2 + \sigma_{32}\hat{n}_3$$

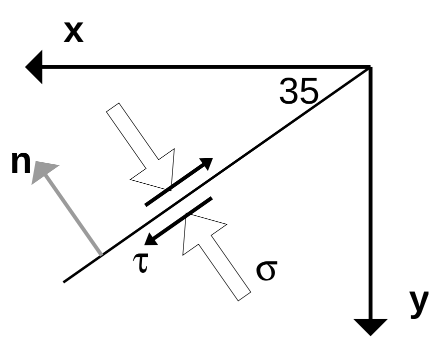
$$T_3 = \sigma_{13}\hat{n}_1 + \sigma_{23}\hat{n}_2 + \sigma_{33}\hat{n}_3$$

note that the order of the i or j are not so important because the stress tensor is symmetric, but in general it is important to keep the i's and j's straight.

Problem 2

[Modified from Turcotte and Schubert, 2-12 (2-24 in version 2)]. The state of stress at a point on a fault plane is $\sigma_{yy} = 150$ MPa, $\sigma_{xx} = 200$ MPa, and $\sigma_{xy} = 0$ MPa. What is the normal traction and shear traction on the fault plane if the fault strikes N-S and dips 35 degrees to the west? Do you expect the fault to slip, given your expectation of the value of the coefficient of friction?

Solution



Following the convention that compression is negative (BEWARE, T&S uses the opposite convention! I will use negative for compression.) the stress tensor is:

$$\sigma_{ij} = \begin{pmatrix} -200 & 0 \\ 0 & -150 \end{pmatrix} \text{MPa}$$

This problem can be solved in a couple of ways. With Mohr's Circles, with rotation matrices and by using the fomulas learned in class. The Mohr's circle is the simplest so I will present that first.

Mohr's Circle Solution

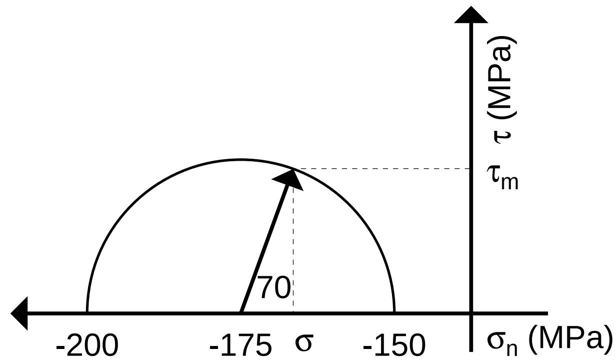
Normal and shear stress are given by Mohr's circle construction by

$$\sigma_n = \sigma_c + \sigma_r \cos(2\theta)$$

$$\tau = \sigma_r \sin(2\theta)$$

$$\sigma_c = \frac{\sigma_{yy} + \sigma_{xx}}{2} = \frac{-150 - 200}{2}\text{MPa} = -175\text{MPa}$$

$$\sigma_r = \frac{\sigma_{yy} - \sigma_{xx}}{2} = \frac{-150 + 200}{2}\text{MPa} = 25\text{MPa}$$



For $\theta = 35°$ we find

$$\sigma_n = -175 + 25\cos(70)\text{MPa} = -166.4\text{MPa}$$

$$\tau = 25\sin(70)\text{MPa} = 23.5\text{MPa}$$

Tensor Rotation Solution

If you had wanted to solve this by rotation, simply rotate the stress tensor ($\sigma' = \alpha\sigma\alpha^T$) by 35° and then $\sigma_{y'y'}$ is the normal stress (traction) and $\sigma_{x'y'}$ is the shear traction. Try it!

The Third Path

First find the traction acting on the fault plane:

$$T_i = \sigma_{ij}\hat{n}_j = \begin{pmatrix} \text{-200 MPa} & 0 \\ 0 & \text{-150 MPa} \end{pmatrix} \begin{pmatrix} \sin 35° \\ -\cos 35° \end{pmatrix} = \begin{pmatrix} -114.7 \\ 122.9 \end{pmatrix} \text{MPa}$$

the normal stress is then found by

$$\sigma_n = T_i\hat{n}_i = -114.7\,\text{MPa}\ \sin 35° - 122.9\,\text{MPa}\ \cos 35° = -166.4\,\text{MPa}.$$

The shear stress is found by

$$\tau = |T_i - \sigma_n \cdot \hat{n}_i| = \left| \begin{pmatrix} -114.7 \\ 122.9 \end{pmatrix} \text{MPa} - \begin{pmatrix} -166.4\ \sin 35° \\ 166.4\ \cos 35° \end{pmatrix} \text{MPa} \right|$$

$$\tau = \left| \begin{pmatrix} -19.3 \\ -13.4 \end{pmatrix} \text{MPa} \right| = \sqrt{(-19.3)^2 + (-13.4)^2}\,\text{MPa} = 23.5\,\text{MPa}$$

The Question of Slipping

For movement to have occurred on the fault we need to satisfy $\tau = \mu\sigma_n$, so we can calculate the critical coefficient of friction, μ_c, needed for slipping at these stresses

$$\mu_c = \frac{\tau}{\sigma_n} = \frac{23.5}{166.4} = 0.14$$

since typical coefficient of friction for rock $(0.6 - 0.85)$ is much greater than the critical coefficient of friction, it is concluded that the fault will not slip.

Alternatively, for a coefficient of friction of 0.6 (typical for rock) and the normal stress determined above, the minimum shear traction, τ_m needed to induce slipping on the fault is

$$\tau_m = (0.6) \times (166.4)\text{MPa} = 99.8\text{MPa}$$

and since $\tau_m > \tau$ the shear stress is not great enough to induce slipping on the fault.

Problem 3

For practice in Einstein summation notation, expand the following expressions using the Kronecker delta, simplify and evaluate the numerical values, where possible (explain what you are doing):

a) δ_{ii}

b) $\delta_{ij}\delta_{ij}$

c) $\delta_{ij}\delta_{jk}$

d) $\delta_{ij}\delta_{jk}\delta_{kl}$

e) $\delta_{ij}A_{ik}$

Solution

To refresh everyone's memory $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ and repeated indices implies a summation over the degrees of freedom, in this problem we will use three degrees of freedom in a three-dimension space.

a) δ_{ii}

The summation is over the index i,

$$\delta_{ii} = \sum_{i=1}^{3} \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

b) $\delta_{ij}\delta_{ij}$

The summation is over both i and j,

$$\delta_{ij}\delta_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\delta_{ij} = \sum_{i=1}^{3} \delta_{i1}\delta_{i1} + \delta_{i2}\delta_{i2} + \delta_{i3}\delta_{i3}$$

and expanding $i = 1$ fully $\delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13} = 1 + 0 + 0 = 1$, and showing only the non-zero terms in the full expansion

$$\delta_{ij}\delta_{ij} = \delta_{11}\delta_{11} + \delta_{22}\delta_{22} + \delta_{33}\delta_{33} = 1 + 1 + 1 = 3$$

c) $\delta_{ij}\delta_{jk}$

The summation is over j and we have 3×3 equations for $i = 1, 2, 3$ and $k = 1, 2, 3$.

$$\delta_{ij}\delta_{jk} = \sum_{j=1}^{3} \delta_{ij}\delta_{jk} = \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k}$$

and note that the only non-zero terms will $i = k$, for example for $i = 1$ and $k = 1$ the sum will be $\delta_{11}\delta_{11} + 0 \cdot 0 + 0 \cdot 0 = 1$ and for $i = 1$ and $k = 2$ the sum will be $1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0$. Therefore we can simplify the above as

$$\delta_{ij}\delta_{jk} = \delta_{ik}$$

and evaluating the 9 equations ($i = 1, 2, 3$ and $k = 1, 2, 3$) and displaying the answers as a matrix we have

$\delta_{ij}\delta_{jk} = \delta_{ik} \Rightarrow$

	$k=1$	$k=2$	$k=3$
$i=1$	1	0	0
$i=2$	0	1	0
$i=3$	0	0	1

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d) $\delta_{ij}\delta_{jk}\delta_{kl}$

This is similar to part c, but here the summation is over j and k where there are again 9 equations for i and l equal to 1, 2 and 3. We can reduce this by applying the relation that we discovered in part c twice

$$\delta_{ij}\delta_{jk}\delta_{kl} = \delta_{ij}\delta_{jl} = \delta_{il}.$$

e) $\delta_{ij}A_{ik}$

Here we sum over i and again we will have 9 equations as in the previous two problems.

$$\delta_{ij}A_{ik} = \sum_{i=1}^{3} \delta_{ij}A_{ik} = \delta_{1j}A_{1k} + \delta_{2j}A_{2k} + \delta_{3i}A_{3k}$$

And as an example, the equation when $j = 1$ and $k = 2$ the sum reduces to

$$\delta_{11}A_{12} + \delta_{21}A_{22} + \delta_{31}A_{32} = 1 \cdot A_{12} + 0 \cdot A_{22} + 0 \cdot A_{32} = A_{12} = A_{jk}$$

You can easily verify that this holds in general by expanding all nine equations out. We can evaluate and display the nine equations again as a matrix

$\delta_{ij}A_{ik} = A_{jk} \Rightarrow$

	$k=1$	$k=2$	$k=3$
$j=1$	A_{11}	A_{12}	A_{13}
$j=2$	A_{21}	A_{22}	A_{23}
$j=3$	A_{31}	A_{32}	A_{33}

$$\Rightarrow \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

Problem 4

For the stress tensor

$$\sigma_{ij} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

a) Find the principal stresses and directions. (Use a right-handed coordinate system)

b) Compare the three Invariants for the original stress tensor and the principal stress tensor.

c) Find the deviatoric stress tensor σ_{ij}^{dev}

d) Find the principal stresses and directions of the deviatoric stress tensor. Find the Invariants of the deviatoric stress tensor.

e) What are the relations of the principal stresses, directions, shears, etc.= for these two tensors (the complete and deviatoric tensors investigated above)? Can you say anything in general about these relations for an arbitrary stress tensor and its deviator?

f) Construct Mohr's circle for σ_{ij}.

g) Construct Mohr's circle for σ_{ij}^{dev}.

h) What is the relation between the maximum shear for the two tensors?

Solution

a) Finding the principal stresses and directions is done in the same way as diagonalizing a matrix or finding the principal coordinate system (*aka* the *eigenbasis*) — in fact these are all the same problem. In essence, we want to find the right-handed coordinate system, expressed in terms of the given coordinate system, in which only normal stresses act along each of the basis vectors. There is a lot of linear algebra hidden in here, but I will just state that given any stress tensor, you can find a pricipal coordinate system. Mathematically, we can express the above statement like this:

$$\sigma_{ij} \cdot \hat{n}_j = \lambda \hat{n}_j \tag{1}$$

which just says that the traction on the $\hat{n}_j$ plane (remember we define our planes by use of the vector normal to it) is along the normal vector (*i.e.* the stress is normal to the plane). We rewrite equation 1 as

$$(\sigma_{ij} - \lambda \cdot \delta_{ij}) \cdot \hat{n}_j = 0 \tag{2}$$

and the λ's and $\hat{n}_j$'s for which this is true are the eigenvalues (principal stresses) and the eigenvectors (principal coordinate systems), respectively. There will be three eigenvalues and associated eigenvecors which will solve equation 2, not neccisarily unique. Two further constraints are placed on the $\hat{n}_j$'s, they must be normalized (unit length) and they must for a right-handed coordinate system. We can express these two conditions as

normalized: $|\hat{n}_j| = \hat{n}_j \cdot \hat{n}_j = \sqrt{n_1^2 + n_2^2 + n_3^2} = 1$

right-handedness: $\hat{n}_i = \hat{n}_j \times \hat{n}_k$

We proceed by finding the eigenvalues (principal stresses), which are the λ's which solve equation 2, we denote them in a matrix as:

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

and are found by solving for the roots of the characteristic polynomial

$$\det\left(\sigma_{ij} - \lambda\delta_{ij}\right) = 0$$

which can be simplified by use of the Invariants or solved directly. I will solve this directly, and you can refer to your notes for how the characteristic polynomial simplifies with the Invariants.

$$\begin{vmatrix} (1-\lambda) & 1 & 0 \\ 1 & (1-\lambda) & 0 \\ 0 & 0 & (2-\lambda) \end{vmatrix} = (2-\lambda)\cdot\left[(\lambda-1)^2 - 1\right] = (2-\lambda)(\lambda-2)\lambda = 0$$

The solutions to this characteristic polynomial are $\lambda = 2,\ 2$ and 0, these are the principal stresses. Following the convention $\sigma_1 > \sigma_2 > \sigma_3$, $\sigma_1 = \sigma_2 = 2$ and $\sigma_3 = 0$.

We then find the associated eigenvectors (principal directions), by solving for a $\hat{n}_j$ for each λ in equation 2. For simplicity, I am going to call the components of the eigenvectors x_j.

$$\text{For } \lambda = 0, \begin{pmatrix} \text{1-0} & 1 & 0 \\ 1 & \text{1-0} & 0 \\ 0 & 0 & \text{2-0} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 + x_2 &= 0 \\ x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= s \\ x_2 &= -s \\ x_3 &= 0 \end{aligned}$$

where s is an undetermined variable. We now enforce the normalization constraint

$$\sqrt{s^2 + (-s)^2 + 0^2} = \sqrt{2s^2} = \sqrt{2}\, s = 1$$

so $s = \frac{1}{\sqrt{2}}$ (note since this equation involves squaring, the sign is ambiguous, so we just choose one, in this case positive), and the eigenvector associated with $\sigma_3 = 0$ is

$$\hat{n}_j^{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

We then find the eigenvectors associated with σ_1 and σ_2 in the same way:

For $\lambda = 2$, $\begin{pmatrix} 1-2 & 1 & 0 \\ 1 & 1-2 & 0 \\ 0 & 0 & 2-2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \left.\begin{array}{rcl} x_1 - x_2 & = & 0 \\ 0 \cdot x_3 & = & 0 \end{array}\right\} \Rightarrow \begin{array}{rcl} x_1 & = & s \\ x_2 & = & s \\ x_3 & = & t \end{array}$$

where s and t are undetermined variables. This gives us two vectors:

$$\hat{n}_j^{(1)} = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \text{ and } \hat{n}_j^{(2)} = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix}.$$

We now enforce the normalization constraint on $\hat{n}_j^{(2)}$

$$\sqrt{s^2 + s^2 + 0^2} = \sqrt{2s^2} = \sqrt{2}\, s = 1$$

so $s = \frac{1}{\sqrt{2}}$ (again note the sign ambiguity, we choose positive), and the eigenvector associated with $\sigma_2 = 2$ is

$$\hat{n}_j^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Now, the normalization constraint on $\hat{n}_j^{(1)}$ is trivial, and it tells us that $t = \pm 1$, so to resolve the ambiguity we now utalize the right-handedness constraint:

$$\hat{n}_i^{(1)} = \left(\hat{n}_j^{(2)} \times \hat{n}_k^{(3)}\right) = \frac{1}{\sqrt{2}} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \hat{\imath}\cdot 0 - \hat{\jmath}\cdot 0 + \hat{k}\cdot(-1-1)\cdot\frac{1}{\sqrt{2}} = \begin{pmatrix} 0 \\ 0 \\ -2/\sqrt{2} \end{pmatrix}$$

so we find, subject to the normalization constraint above,

$$\hat{n}_j^{(1)} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

We can then put this all together as

$$\sigma_{nn}^{\text{principal}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \hat{n}_{j(n)} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ -\sqrt{2} & 0 & 0 \end{pmatrix},$$

where the first matrix is the principal stress tensor, and the columns of the second matrix define the principal frame, given in terms of the original coordinate system. Note that the final solution to this problem is non-unique in terms of the signs of the eigenvectors, which is to say it is non-unique in terms of $\frac{\pi}{2}$ rotations about the axes.

b)

For the origninal stress tensor:

$$\sigma_{ij} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\mathbf{I}_1 = \sigma_{ii} = 1 + 1 + 2 = 4$$

$$\mathbf{I}_2 = \frac{1}{2} \cdot (\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}) = \frac{1}{2} \cdot \left[\sigma_{11}^2 + 2\sigma_{12}^2 + \sigma_{22}^2 + \sigma_{33}^2 - (\sigma_{11} + \sigma_{22} + \sigma_{33})^2\right] = -4$$

$$\mathbf{I}_3 = |\sigma_{ij}| = \det(\sigma_{ij}) = 2(1 - 1) = 0$$

For the principal stress tensor:

$$\sigma_{ij} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{I}_1 = \sigma_{ii} = 2 + 2 + 0 = 4$$

$$\mathbf{I}_2 = \frac{1}{2} \cdot (\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}) = -\sigma_{11}\sigma_{22} = -4$$

$$\mathbf{I}_3 = |\sigma_{ij}| = \det(\sigma_{ij}) = 0$$

The Invariants of the original stress tensor and the principal stress tensor are the same — they are invariant under coordinate system rotation.

c)
The deviatoric stress tensor is found by $\sigma_{ij}^{dev} = \sigma_{ij} - \frac{\sigma_{ii}}{3} \cdot \delta_{ij}$, and is

$$\sigma_{ij}^{dev} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \frac{1}{3} \cdot \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -1/3 & 1 & 0 \\ 1 & -1/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}$$

d) Following the proceedure in part a, we find the eigenvalues (principal stresses) and the eigenvectors (principal coordinates) as

$$\sigma_{nn}^{\text{principal}} = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & -4/3 \end{pmatrix} \text{ and } \hat{n}_{j(n)} \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ -\sqrt{2} & 0 & 0 \end{pmatrix},$$

The Invariants are

$$\mathbf{I}_1^{dev} = 0$$
$$\mathbf{I}_2^{dev} = \frac{4}{3}$$
$$\mathbf{I}_3^{dev} = \frac{-16}{27}$$

e) The definition of the deviatoric stress tensor is $\sigma_{ij}^{dev} = \sigma_{ij} - \frac{\sigma_{kk}}{3} \cdot \delta_{ij}$, which simply says that the deviatoric stress tensor is the stress tensor minus the average of the normal stresses. So we decrease the principal stresses by the same amount. Since the shear stresses stay the same, while the normal stresses all decrease by the same amount, the principal coordinates stay the same.

The relationship between the Invariants of the stress tensor and its deviator can be found by (keep track of the indecies):

$$\mathbf{I}_1^{dev} = \sigma_{ii}^{dev} = \sigma_{ii} - \frac{\sigma_{kk}}{3}\delta_{ii} = \sigma_{ii} - \frac{\sigma_{kk}}{3} 3 = 0$$

$$\mathbf{I}_3^{dev} = \frac{1}{2}\left[\sigma_{ij}^{dev}\sigma_{ij}^{2} - \sigma_{ii}^{dev}\sigma_{jj}^{dev}\right]$$

$$= \frac{1}{2}\left[\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right) - \left(\sigma_{ii} - \frac{1}{3}\sigma_{kk}\delta_{ii}\right)\left(\sigma_{jj} - \frac{1}{3}\sigma_{kk}\delta_{jj}\right)\right]$$

note that $\delta_{ii} = 3$, $\delta_{ij}\delta_{ij} = 3$, and $\sigma_{ij}\delta_{ij} = \sigma_{ii}$, and expanding the multiplied terms

$$= \frac{1}{2}\left[\sigma_{ij}\sigma_{ij} - \frac{2}{3}\sigma_{kk}\sigma_{jj} + \frac{3}{9}\sigma_{kk}\sigma_{kk} - \sigma_{ii}\sigma_{jj} + 2\sigma_{kk}\sigma_{jj} - \sigma_{kk}\sigma_{jj}\right]$$

$$= \frac{1}{2}\left[\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj} + \sigma_{kk}\sigma_{jj}\left(\frac{-1}{3} + \frac{6}{3} - \frac{2}{3}\right)\right]$$

$$= \frac{1}{2}\left(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}\right) + \frac{1}{2}\frac{2}{3}\sigma_{kk}\sigma_{jj}$$

$$\Rightarrow \mathbf{I}_2^{dev} = \mathbf{I}_2 + \frac{1}{3}\mathbf{I}_1^2$$

For $\mathbf{I}_3^{dev}$ I will derive this for the principal coordinate system, which no loss of generality, setting $p = \frac{-1}{3}\sigma_{kk} = \frac{-1}{3}\mathbf{I}_1$:

$$\mathbf{I}_3^{dev} = \det\left(\sigma_{ij}^{dev}\right) = \sigma_{11}^{dev}\sigma_{22}^{dev}\sigma_{33}^{dev} = \left(\sigma_{11} + p\right)\left(\sigma_{22} + p\right)\left(\sigma_{33} + p\right)$$

$$= \sigma_{11}\sigma_{22}\sigma_{33} + p\left[\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} + p\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right) + p^2\right]$$

$$= \mathbf{I}_3 + p\left[-\mathbf{I}_2 - 3p^2 + p^2\right] = \mathbf{I}_2 - \left(\frac{-1}{3}\mathbf{I}_1\right)\left(-\mathbf{I}_2\right) - 2\left(\frac{-1}{3}\mathbf{I}_1\right)^3$$

$$\Rightarrow \mathbf{I}_3^{dev} = \mathbf{I}_3 + \frac{1}{3}\mathbf{I}_1\mathbf{I}_2 + \frac{2}{27}\mathbf{I}_1^3$$

f) Mohr's circle is given by

$$\sigma_n = \sigma_c + \sigma_r \cos 2\theta$$

$$\tau = \sigma_r \sin 2\theta$$

where $\sigma_r = \frac{\sigma_1 - \sigma_2}{2}, \sigma_r = \frac{\sigma_1 + \sigma_2}{2}$ and $\sigma_{1,2}$ are the principal stresses ($\sigma_1 > \sigma_2$). For σ_{ij}

$$\sigma_2 = 0, \quad \sigma_1 = 2$$

which gives

$$\sigma_r = \sigma_c = 1.$$

The plot is given below.

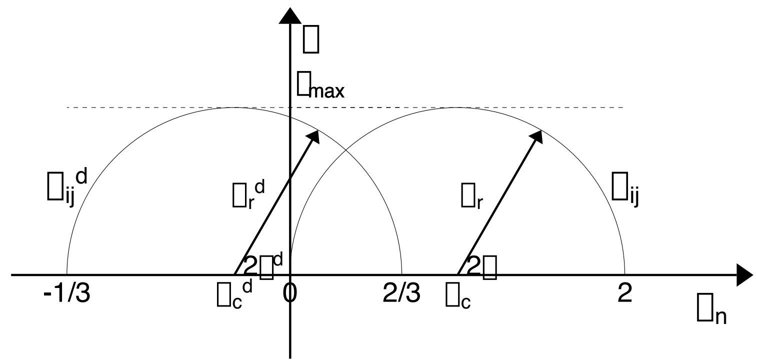
g) For σ_{ij}^{dev}

$$\sigma_2 = \frac{-4}{3}, \quad \sigma_1 = \frac{2}{3}$$

which gives

$$\sigma_r = 1 \text{ and } \sigma_c = \frac{-1}{3}.$$

The plot of the Mohr's circle for the full stress tensor (σ_{ij}) and its deviatoric component (σ_{ij}^{dev}, labelled with a superscript d in the plot) is below.



h) The maximum shear stress occurs when $\sin 2\theta = 0$, which gives $2\theta = \frac{\pi}{2}$. (This is also apparent from examining the plot of Mohr's circle.) For $2\theta = \frac{\pi}{2}$, $\tau = \sigma_r$ is the maximum that the shear stress can be. Since $\sigma_r = 1$ for the two tensors (σ_{ij} and σ_{ij}^{dev}) it follows that the maximum shear stress is the same for both of the tensors.