Problem Solution

For the small strains involved in most elastic problems, the mathematics describing stresses and strains are linear. This means that the solution to a more complicated problem can be found by breaking the problem into smaller, simpler pieces and adding the solutions together. As an example, you can determine the relationship between the bulk modulus and other elastic moduli such as Young's modulus E, the Lamé parameters λ and μ (where μ = G, the shear modulus), and Poisson's ratio ν in the following way:

i) First consider the stresses and strains involved in uniaxial stress along the x_1 axis. That is, consider the experiment used to define Young's modulus E: for $\sigma_{11} = \sigma_0$ and all other $\sigma_{ij} = 0$, write each component of the strain tensor, first in terms of λ and μ, then in terms of E and ν.

ii) Next, consider uniaxial stress σ_0 along the x_2 direction. Write each component of the strain tensor, first in terms of λ and μ, then in terms of E and ν.

iii) Next, consider uniaxial stress σ_0 along the x_3 direction. Write each component of the strain tensor, first in terms of λ and μ, then in terms of E and ν.

iv) Finally, add these three solutions together to determine the total strains involved when a stress σ_0 is applied along each axis at the same time. From these relations between stress and strain, determine the bulk modulus K, first in terms of λ and μ, then in terms of E and ν.

Solution:

For an isotropic media we can write down the expression for the components of the stress tensor as: $\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$, where λ and μ are the Lame parameters,

and for the strain tensor: $e_{ij} = \dfrac{1}{2\mu}\left(\tau_{ij} - \dfrac{\lambda}{2\mu + 3\lambda}\delta_{ij}\tau_{kk}\right)$

i) Let's consider the stresses and strains involved for uniaxial stress along the x_1 axis:

$$\sigma_{11} = \sigma_0 \text{ and } \sigma_{ij}\Big|_{i,j\neq 1,1} = 0$$

The components of the strain tensor in terms of λ and μ are:

$$e_{11} = \frac{\mu + \lambda}{\mu(3\lambda + 2\mu)}\sigma_0 \quad \text{and} \quad e_{22} = e_{33} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\sigma_0$$

then in terms of E and ν:

$$e_{11} = \frac{1}{E}\sigma_0 \text{ and since } e_{22} = e_{33} \propto -e_{11} \propto -\frac{1}{E}\sigma_0 \text{ then } e_{22} = e_{33} = -\frac{\nu}{E}\sigma_0$$

ii) Along the x_2 direction: $\sigma_{22} = \sigma_0$ and $\sigma_{ij}\big|_{i,j\neq 2,2} = 0$

The components of the strain tensor in terms of λ and μ:

$$e_{22} = \frac{\mu+\lambda}{\mu(3\lambda+2\mu)}\sigma_0 \quad \text{and} \quad e_{11} = e_{33} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_0$$

then in terms of E and ν:

$$e_{22} = \frac{1}{E}\sigma_0 \text{ and } e_{11} = e_{33} = -\frac{\nu}{E}\sigma_0$$

iii) Along the x_3 direction: $\sigma_{33} = \sigma_0$ and $\sigma_{ij}\big|_{i,j\neq 3,3} = 0$

The components of the strain tensor in terms of λ and μ:

$$e_{33} = \frac{\mu+\lambda}{\mu(3\lambda+2\mu)}\sigma_0 \quad \text{and} \quad e_{11} = e_{22} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_0$$

then in terms of E and ν:

$$e_{33} = \frac{1}{E}\sigma_0 \text{ and } e_{11} = e_{22} = -\frac{\nu}{E}\sigma_0$$

iv) Let's add these three solutions together and determine the total strains involved when a stress σ_0 is applied along each axis at the same time: $\sigma_{ij} = \sigma_0 \cdot \delta_{ij}$

$$e_{ij} = \frac{1-2\nu}{E}\sigma_0 \cdot \delta_{ij} = \frac{1}{3\lambda+2\mu}\sigma_0 \cdot \delta_{ij}$$

From these relations between stress and strain we can determine the bulk modulus K :

$$K \equiv -\frac{P}{\Delta V/V} = \frac{\sigma_0}{e_{kk}} \text{ in terms of } \lambda \text{ and } \mu$$

and in terms of E and ν:

$$K = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)}$$

Problem Solution

The relationship between stress and strain for a simple isotropic elastic material is:

$$\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

(λ and μ are constants, the Lame parameters, and are called "moduli")

It is sometimes useful to present ε_{ij} rather than σ_{ij} , as the independent variable. One way of determining the equation for ε_{ij} in terms of σ_{ij} and σ_{kk} is to first solve the equation above for ε_{kk} in terms of σ_{kk}, then substitute this value for ε_{kk} and solve for ε_{ij} .
First, determine the expression for ε_{kk} in terms of σ_{kk}. Then, write an equation equivalent to the one above, but with ε_{ij} on the left side and stresses on the right hand side. (The constants multiplying the stresses are called "compliances".)

Solution:

Let's express σ_{kk} explicitly:

$$\tau_{kk} = \tau_{11} + \tau_{22} + \tau_{33}$$

and substitute stresses τ_{11} τ_{22} τ_{33} in terms of the strains using the relationship: $\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$. Then we get expression for the trace of stress tensor:

$$\tau_{kk} = 3\lambda e_{kk} + 2\mu\left(e_{11} + e_{22} + e_{33}\right) = 3\lambda e_{kk} + 2\mu e_{kk},$$

which can be rewritten to the expression for the trace of the strain tensor (dilatation):

$$e_{kk} = \frac{\tau_{kk}}{3\lambda + 2\mu}.$$

Substituting the dilatation into the expression for the stress tensor, $\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$, we rewrite it with respect to the strain tensor:

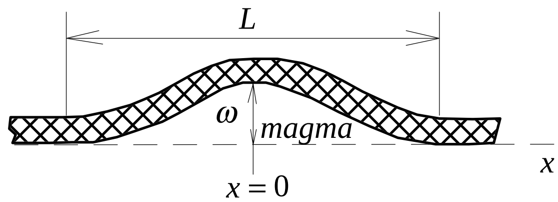
$$e_{ij} = \frac{1}{2\mu}\left(\tau_{ij} - \frac{\lambda}{2\mu + 3\lambda}\tau_{kk}\delta_{ij}\right)$$

Problem Solution

Turcotte & Schubert, Problem 3-17 (See pages 119-120)
Determine the bending moment in the overburden above the idealized two-dimensional laccolith as a function of x. Where is M a maximum? What is the value of M_{max}?

Solution:

An idealized two-dimensional laccolith can be viewed as following:



with an assumptions that its deflection is much smaller than its horizontal scale:

$$\omega << L$$

The bending moment in the overburden above this idealized two-dimensional laccolith is defined as:

$$M = -\mathrm{D}\frac{d^2\omega}{dx^2},$$

where flexural rigidity: $\mathrm{D} = \dfrac{Eh^3}{12(1-\nu^2)}$,

deflection of the plate: $\omega(x) = \dfrac{\mathrm{q}}{24\mathrm{D}}\left(x^4 - \dfrac{L^2}{2}x^2 + \dfrac{L^4}{16}\right) = \omega_0\left(1 - 8\left(\dfrac{x}{L}\right)^2 + 16\left(\dfrac{x}{L}\right)^4\right)$,

with q as an external vertical force acting on the plate, and ω_0 as the deflection at x=0: $\omega(x)\big|_{x=0} = \omega_0 = \dfrac{\mathrm{q}L^4}{384\mathrm{D}}$.

We take the double derivatives of the deflection and write the bending moment expression as:

$$M = \frac{16\omega_0 D}{L^2}\left(1 - 12\left(\frac{x}{L}\right)^2\right) = \frac{qL^2}{24}\left(1 - 12\left(\frac{x}{L}\right)^2\right).$$

Mathematically, an extreme (minimum or maximum) of a function can be found as the place where its derivative with respect to x equals zero, that is where: $M'_x = 0$. But we have to calculate also the bending moment at the edges of the overburden (just in case if it is greater than the mathematically found extremes).

The derivative of the bending moment is zero when $x=0$; there the bending moment is:

$$M_{x=0} = \frac{16\omega_0 D}{L^2} = \frac{qL^2}{24}$$

The bending moment at the edges of the overburden ($x=\pm L/2$) is:

$$M_{x=\pm L/2} = -\frac{32\omega_0 D}{L^2} = -\frac{qL^2}{12},$$

which is twice as big as at the center.

Therefore, the maximum of the bending moment in the overburden above the idealized two-dimensional laccolith is located at the edges: *x=±L/2.*

It equals to: $M_{\max} = -\frac{32\omega_0 D}{L^2} = -\frac{qL^2}{12}$

Problem Solution

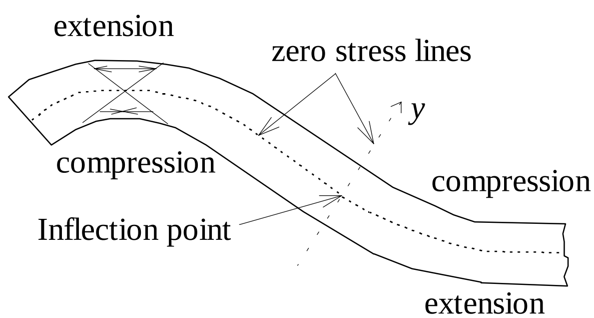
Problem 3-18, Turcotte & Schubert:

Calculate the fiber stress in the stratum overlying the two-dimensional laccolith as a function of *y* (distance from the centerline of the layer) and *x*.

If dikes tend to form where tension is greatest in the base of the stratum forming the roof of a laccolith, where would you expect dikes to occur for the two-dimensional laccolith?

Solution:

The state of the fiber stress in the stratum is shown schematically in the picture:



The fiber stress is related to the strain as: $\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx}$, where strain depends on both coordinates, *x* and *y*: $\varepsilon_{xx} = -y\frac{d^2\omega}{dx^2}$ through the deflection of the laccolith, ω.

The deflection as a function of *x* is:

$$\omega(x) = \frac{\mathrm{q}}{24\mathrm{D}}\left(x^4 - \frac{L^2}{2}x^2 + \frac{L^4}{16}\right) = \omega_0\left(1 - 8\left(\frac{x}{L}\right)^2 + 16\left(\frac{x}{L}\right)^4\right),$$

where flexural rigidity: $\mathrm{D} = \frac{Eh^3}{12(1-\nu^2)}$; q is an external vertical force acting on the plate, and ω_0 is the deflection at *x*=0: $\omega(x)\big|_{x=0} = \omega_0 = \frac{\mathrm{q}L^4}{384\mathrm{D}}$.

We take the double derivatives of the deflection and write the strain as:

$$\varepsilon_{xx} = y\frac{16\omega_0}{L^2}\left(1 - 12\left(\frac{x}{L}\right)^2\right) = y\frac{qL^2}{24D}\left(1 - 12\left(\frac{x}{L}\right)^2\right).$$

Substituting expression for the flexural rigidity, we obtain the fiber stress in the stratum overlying the two-dimensional laccolith as a function of *y* and *x*:

$$\sigma_{xx} = \frac{qL^2}{2h^3}\, y\left(1 - 12\left(\frac{x}{L}\right)^2\right)$$

If dikes tend to form where tension is greatest, let's determine where the fiber stress is maximal in the two-dimensional laccolith.

An extreme (minimum or maximum) of a function can be found as the place where its derivative equals zero. Since in our model the horizontal dimension of the stratum has finite length, we have to calculate also the stress at the edges.

The derivative of the fiber stress is zero at $x=0$; where the stress is:

$$\sigma_{xx}\Big|_{x=0,y=h/2} = \frac{q}{4}\left(\frac{L}{h}\right)^2,$$

here we assumed $y=h/2$.

The fiber stress at the edges of the stratum ($x=\pm L/2$) is:

$$\sigma_{xx}\Big|_{x=\pm L/2,y=h/2} = -\frac{q}{2}\left(\frac{L}{h}\right)^2.$$

which is twice as big as at the center.

Therefore, the maximum fiber stress in the two-dimensional laccolith is at its edges: $x=\pm L/2$ and it equals to:

$$\sigma_{xx}\Big|_{\max} = -\frac{q}{2}\left(\frac{L}{h}\right)^2.$$

If the dikes tend to form where tension is greatest, we would expect them to occur in the base of the stratum where it has a concave upwards shape.

Problem Solution

Problem 3-19, part b) only, Turcotte & Schubert. [Note that part a) of this problem discusses only one of multiple maxima in the bending moment. There are other maxima. You should identify the one with the largest bending stress.]

(**a**) Consider a lithospheric plate under a line load.

(**b**) Refraction studies show that the Moho is depressed about 10 km beneath the center of the Hawaiian Islands. Assuming that this is the value of ω_o and that h = 34 km, E = 70 Gpa, ν = 0.25, ρ_m-ρ_w = 2300 kg/m^3, and g = 10 m/s^2, determine the maximum bending stress in the lithosphere.

Solution:

The bending stress in the lithosphere is related to the strain as: $\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx}$, where strain depends on both coordinates, x and y: $\varepsilon_{xx} = -y\frac{d^2\omega}{dx^2}$ through the deflection of the plate, ω.

As we derived in the class, the plate deflection under a line load is a function of x:

$$\omega(x) = \omega_0\left(\cos\frac{x}{\alpha} + \sin\frac{x}{\alpha}\right)\exp\left\{-\frac{x}{\alpha}\right\}.$$

Therefore, the bending stress in the lithosphere is:

$$\sigma_{xx} = -y\frac{E}{1-\nu^2}\frac{2\omega_0}{\alpha^2}\left(\sin\frac{x}{\alpha} - \cos\frac{x}{\alpha}\right)\exp\left\{-\frac{x}{\alpha}\right\}.$$

To determine where the bending stress in the lithosphere reaches maxima in x-direction, we need to set its x-derivative equal to zero: $\frac{\partial\sigma_{xx}}{\partial x} = 0$. Solving the last equation, we obtain the x-coordinates where the bending stress is maximum: $\frac{x}{\alpha} = \pm\frac{\pi}{2}$.

Since x and the flexural parameter α must be positive, we substitute back into the equation for the stress only positive solution. Then the stress becomes:

$$\sigma_{xx} = y\frac{E}{1-\nu^2}\frac{2\omega_0}{\alpha^2}\exp\left\{-\frac{\pi}{2}\right\}.$$

To calculate the numerical value, we use the definitions for the flexural parameter: $\alpha^4 = \frac{4D}{(\rho_m - \rho_w)g}$ and the flexural rigidity: $\mathrm{D} = \frac{Eh^3}{12(1-\nu^2)}$, which gives:

$$\alpha^4 = \frac{Eh^3}{3(1-\nu^2)(\rho_m - \rho_w)g}.$$

We assume that the stress reaches maximum at $y=\pm h/2$ and substitute all last formulas into the expression for the stress. Then the maximum bending stress in the lithosphere becomes:

$$\sigma_{xx}\big|_{\max} = \omega_0 \exp\left\{-\frac{\pi}{2}\right\}\sqrt{\frac{3Eg(\rho_m - \rho_w)}{h(1-\nu^2)}} \approx 7.5 \text{ kbars}.$$