Problem 1

In the x_1, x_2, x_3 coordinate system, the stress tensor σ_{ij} is given by:

$$\sigma_{ij} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & -\sqrt{3} \\ 0 & -\sqrt{3} & 9 \end{pmatrix}$$

Consider the new ("primed") coordinate system obtained by rotating by 60° about the x_1 axis.

a) Determine σ'_{ij}.

b) Determine, compare and contrast the principal stresses and principal directions of σ_{ij} and σ'_{ij}.

Solution

a) The rotation matrix about the x_1 axis is

$$\alpha_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(60) & \sin(60) \\ 0 & -\sin(60) & \cos(60) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

and the rotated stress tensor is found by $\sigma'_{ij} = \alpha_{ik}\sigma_{kl}\alpha^t_{lj}$

$$\sigma'_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & -\sqrt{3} \\ 0 & -\sqrt{3} & 9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$\sigma'_{ij} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 27/4 & 5\sqrt{3}/4 \\ 0 & 5\sqrt{3}/4 & 33/4 \end{pmatrix}$$

b) First find the principal stresses and coordinates for σ_{ij} (for this problem you are encouraged to use a computer mathematic program or a calculator which solves the eigenbasis problem):

$$\begin{vmatrix} (3-\lambda) & 0 & 0 \\ 0 & (6-\lambda) & -\sqrt{3} \\ 0 & -\sqrt{3} & (9-\lambda) \end{vmatrix} = (\lambda - 3) \cdot \left[\lambda^2 - 15\lambda + 51\right] = 0$$

The solutions to this characteristic polynomial are $\lambda = 3,$ and $\frac{15\pm\sqrt{21}}{2}$, these are the principal stresses.

We then find the associated eigenvectors as in Problem 1. For $\sigma_1 = \frac{15+\sqrt{21}}{2} \approx 9.8$,

$$\hat{n}^{(1)}_j \approx \begin{pmatrix} 0 \\ 0.4 \\ -0.9 \end{pmatrix}$$

For $\sigma_2 = \frac{15-\sqrt{21}}{2} \approx 5.2$,

$$\hat{n}^{(2)}_j \approx \begin{pmatrix} 0 \\ 0.9 \\ 0.4 \end{pmatrix}$$

For $\sigma_3 = 3$,

$$\hat{n}^{(3)}_j = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We can then put this all together as

$$\sigma^{\text{principal}}_{nn} \approx \begin{pmatrix} 9.8 & 0 & 0 \\ 0 & 5.2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ and } \hat{n}_{j(n)} \approx \begin{pmatrix} 0 & 0 & 1 \\ 0.4 & 0.9 & 0 \\ -0.9 & 0.4 & 0 \end{pmatrix}.$$

We then find the principal stresses and coordinates for σ'_{ij}.

$$\begin{vmatrix} (3-\lambda) & 0 & 0 \\ 0 & (27/4-\lambda) & 5\sqrt{3}/4 \\ 0 & 5\sqrt{3}/4 & (33/4-\lambda) \end{vmatrix} = 0$$

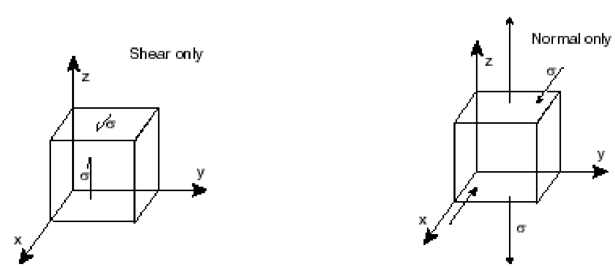
The solutions to this characteristic polynomial are again $\lambda = 3$, and $\frac{15\pm\sqrt{21}}{2}$, these are the principal stresses. The principal stresses and associate coordinates are

$$\sigma_{nn}^{\text{principal}} \approx \begin{pmatrix} 9.8 & 0 & 0 \\ 0 & 5.2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ and } \hat{n}'_{j(n)} \approx \begin{pmatrix} 0 & 0 & 1 \\ -0.6 & 0.8 & 0 \\ -0.8 & -0.6 & 0 \end{pmatrix}.$$

The principal stress for σ_{ij} and σ'_{ij} are the same — this is as expected, the principal stresses do not change under rotation of the stress tensor. Moreover, as you can prove, the principal coordinates of σ'_{ij} are the principal coordinates of σ_{ij} rotated by 60° — this is also as we expect, as the eigenvectors (or principal coordinates) are found in terms of the coordinate system the stress tensor is defined in.

Problem 2

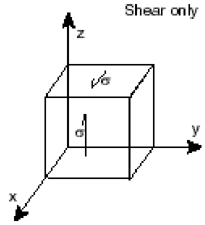
Give the stress tensors, principal stresses, Mohr's circle and maximum shear stresses for the following cases of tractions applied to a unit cube:

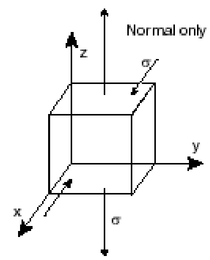


In all cases, the magnitude of the applied traction is σ.

What can you say about the relationship between the stress tensors for the two examples shown?

Solution





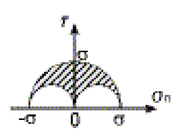
stress tensor

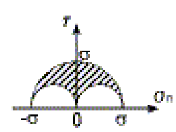
$$\begin{pmatrix} 0 & 0 & \sigma \\ 0 & 0 & 0 \\ \sigma & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} -\sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

principal stresses

$$\{\sigma_1, \sigma_2, \sigma_3\} = \{-\sigma, 0, \sigma\} \qquad \{\sigma_1, \sigma_2, \sigma_3\} = \{-\sigma, 0, \sigma\}$$

Mohr's circle

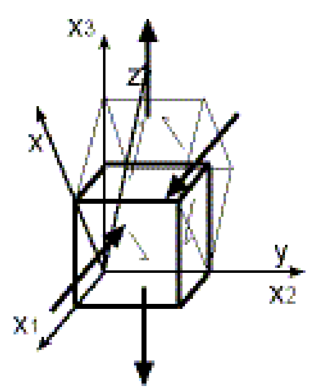




maximum shear stresses

$$\tau_{\max} = \sigma \qquad \tau_{\max} = \sigma$$

Since the principal stresses are the same, the two cubes are in the same state of stress, but the two cubes are described in two different coordinate systems, rotated with respect to each other. The coordinate systems are rotated about the y-axis exactly 45° apart from eachother. If you solved for the principal coordinate systems for either stress tensor above, you would find the coordinate system for the normal only cube.



Problem 3

If there are no body forces, demonstrate whether equilibrium exists, in general, for stresses:

$$\sigma_{xx} = 3x^2 + 4xy - 8y^2$$
$$\sigma_{yy} = 2x^2 + xy + 3y^2$$
$$\sigma_{xy} = \frac{1}{2}x^2 - 6xy - 2y^2$$
$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$$

Solution

The equation of motion is $\sigma_{ij,j} + \rho f_i = \rho a_i$, in the case there are no body forces, f_i, and if equilibrium exists, $a_i = 0$, then $\sigma_{ij,j} = 0$, which is called Newton's Second law for a continuum in class.

Check equilibrium for the above stresses:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \stackrel{?}{=} 0 \;\Rightarrow\; (6x + 4y) + (-6x - 4y) = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \stackrel{?}{=} 0 \;\Rightarrow\; (2\frac{1}{2}x - 6y) + (x + 6y) = 2x \neq 0$$

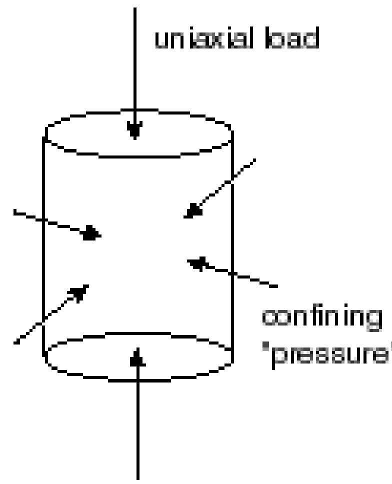
therefore equilibrium is not satisfied along the y-axis. However, in the case when $x = 0$ then equilibrium does exist (*i.e.* equilibrium exists on the $x = 0$ plane).

Problem 4

In a laboratory test, samples of sandstone failed at the conditions:

a) confining "pressure" = 200 bars,
uniaxial load stress = 1,100 bars

b) confining "pessure" = 400 bars,
uniaxial load stress = 1,700 bars



If the strength of the rock can be described by the Navier-Coulomb criterion, what are

1) the intinsic strength S_o,

2) the coefficient of internal friction μ,

3) the orientation of the fractures in each test,

4) the maximum shear stress on any plane in each test, and

5) the shear traction on the failure plane in each test?

Give the answers for dimensional quantities both in these units and in SI units.

Solution

First, we note that the conversion from bars to Passcals (the SI unit for pressure) is 1 bar = 0.1 MPa. The principal stress tensor for the above diagram is

$$\sigma_{ij} = \begin{pmatrix} \sigma_{cp} & 0 & 0 \\ 0 & \sigma_{cp} & 0 \\ 0 & 0 & \sigma_{ul} \end{pmatrix}$$

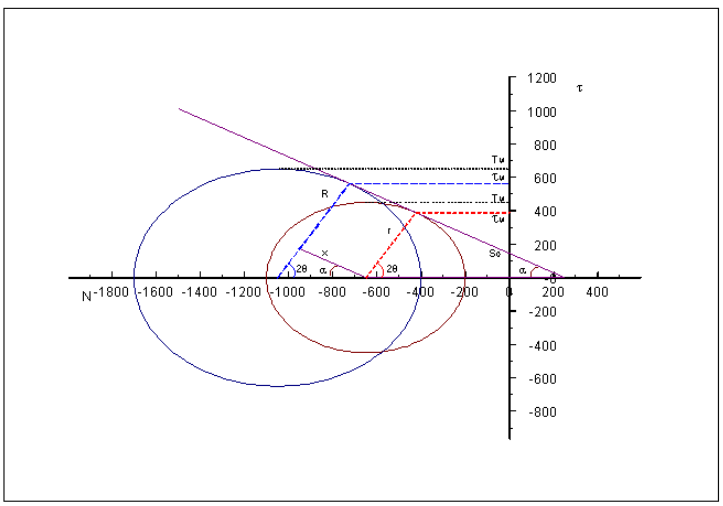
where σ_{cp} and σ_{ul} are the confining pressure and uniaxial load. We now construct the Mohr's circles for cases **a** and **b**:

	a	**b**
$\sigma_r = \frac{\sigma_1 - \sigma_2}{2}$:	$\sigma_r^{\mathbf{a}} = \frac{1100-200}{2}$ bars $= 450$ bars	$\sigma_r^{\mathbf{b}} = \frac{1700-400}{2}$ bars $= 650$ bars
$\sigma_c = \frac{\sigma_1 + \sigma_2}{2}$:	$\sigma_c^{\mathbf{a}} = \frac{1100+200}{2}$bars $= 650$ bars	$\sigma_c^{\mathbf{a}} = \frac{1700+400}{2}$ bars $= 1,050$ bars

and plot the Mohr's circles in stress space with the Navier-Coulomb criterion:

$$|\tau| = \mu\,|\sigma_n| + S_o$$

where we know, since both of the samples failed, that the Navier-Coulomb line must be tangent to both of the Mohr's circles.



where r and R are $\sigma_r^{\mathbf{a}}$ and $\sigma_r^{\mathbf{b}}$, respectively. All of the questions can be answered from the geometry of the plot.

1) We will come back to this one...

2) The coefficient of internal friction μ is the same for both tests, and it is a property of a material and both tests used the same material:

$$\mu = \tan(\alpha) = \frac{\sigma_r^{\mathbf{b}} - \sigma_r^{\mathbf{a}}}{x} = \frac{\sigma_r^{\mathbf{b}} - \sigma_r^{\mathbf{a}}}{\sqrt{(\sigma_c^{\mathbf{b}} - \sigma_c^{\mathbf{a}})^2 - (\sigma_r^{\mathbf{b}} - \sigma_r^{\mathbf{a}})^2}} = \frac{1}{\sqrt{3}} \approx 0.58 \;\Rightarrow\; \alpha = 30^\circ$$

3) The orientation of the fractures is given by θ in the above plot, and is defined as the angle between the normal to the failure plane and the minimum principal stress. The angle θ is the same for both tests since it is the same Navier-Coulomb criterion:

$$2\theta = 90 - \alpha \;\Rightarrow\; \theta = 30^\circ$$

1) The intrinsic strength S_o is also the same for both tests, and is found by:

$$S_o = \tan(\alpha) \cdot \left(\frac{\sigma_r^{\mathbf{a}}}{\sin \alpha} - \sigma_c^{\mathbf{a}} \right) \approx 144 \text{ bars} = 14.4 \text{ MPa}$$

4) The maximum shear stresses are different for the two tests, we get these from the definition of the Mohr's circle:

$$\tau_{\max}^{\mathbf{a}} = \sigma_r^{\mathbf{a}} = 450 \text{ bars} = 45 \text{ MPa}$$

$$\tau_{\max}^{\mathbf{b}} = \sigma_r^{\mathbf{b}} = 650 \text{ bars} = 65 \text{ MPa}$$

5) The shear traction on the failure plane is different in each test. This can be seen geometrically in stress-space, and the shear traction on the failure planes is:

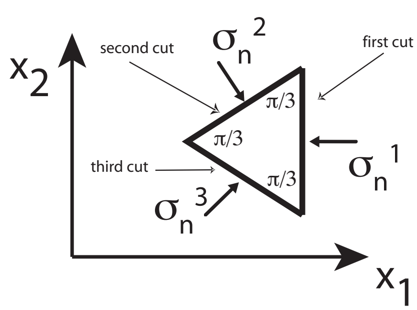
$$\tau^{\mathbf{a}} = \sigma_r^{\mathbf{a}} \cdot \cos(\alpha) \approx 390 \text{ bars} = 39.0 \text{ MPa}$$

$$\tau^{\mathbf{b}} = \sigma_r^{\mathbf{b}} \cdot \cos(\alpha) \approx 563 \text{ bars} = 56.3 \text{ MPa}$$

Problem 5

One way of measuring the state of stress in a rock is to use 3 flat-jacks arranged in a "delta" pattern, with each cut separated by 60°. (The name comes from the resemblance to the Greek character Δ.) The normal traction σ_n across each of the cuts can be measured easily using a flat-jack, as discussed in class. Write the expression for the normal traction across each of the three cuts, σ_n^i, $i = 1, 2, 3$, in terms of the components of the (two-dimensional) stress tensor, σ_{11}, σ_{22}, and σ_{12}.

Solution



The easiest solution to this problem is to go from stress to tractions and then tractions to normal stresses:

$$T_i = \sigma_{ij}\hat{n}_j \text{ and } \sigma_n = T_i\hat{n}_i \;\Rightarrow \sigma_n = \sigma_{ij}\hat{n}_j\hat{n}_i$$

Converting this to vector-matrix, where we take vectors to be columns, the normal traction σ_n acting on the plane with the normal $\hat{n}$ is

$$\sigma_n = \hat{n}^T \sigma \, \hat{n} = \begin{pmatrix} n_1 & n_2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

The normal vectors to the cuts are given by

$$\hat{n}_j^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \hat{n}_j^2 = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \qquad \hat{n}_j^3 = \begin{pmatrix} -1/2 \\ \sqrt{2}/2 \end{pmatrix}$$

and then we find:

$$\sigma_n^1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sigma_{11}$$

$$\sigma_n^2 = \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \cdot \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} = \frac{\sigma_{11} - 2\sqrt{3}\,\sigma_{12} + 3\sigma_{22}}{4}$$

$$\sigma_n^3 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \cdot \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix} = \frac{\sigma_{11} + 2\sqrt{3}\,\sigma_{12} + 3\sigma_{22}}{4}$$

then we can solve for the components of the stress tensor in terms of the observable normal stresses σ_n^k

$$\sigma_{11} = \sigma_n^1$$

$$\sigma_{12} = \frac{2\,(\sigma_n^2 + \sigma_n^3) - \sigma_n^1}{3}$$

$$\sigma_{22} = \frac{\sigma_n^3 - \sigma_n^2}{\sqrt{3}}$$