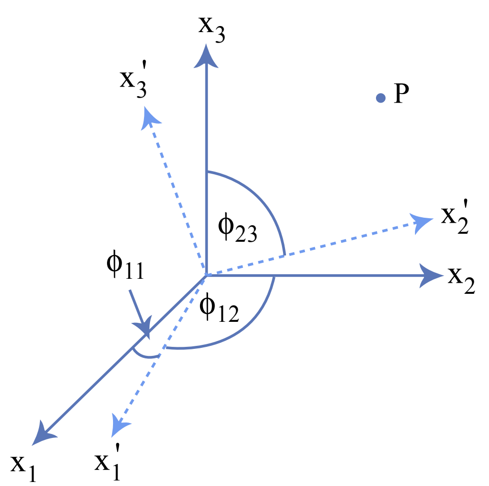
12.005 Lecture Notes 5

Quantities in Different Coordinate Systems

How to express quantities in different coordinate systems?



Direction Cosines

Axis

	x_1	x_2	x_3
x_1'	α_{11}	α_{12}	α_{13}
x_2'	α_{21}	α_{22}	α_{23}
x_3'	α_{31}	α_{32}	α_{33}

Figure 5.1
Figure by MIT OCW.

Direction cosine ij is cosine of angle ϕ_{ij} between primed axis i and unprimed axis j. If $\hat{x}_i'$ and $\hat{x}_j$ represent unit vectors that are the axes of two coordinate systems with the same origin, they are related by the equation

$$\hat{x}_i' = \sum_{j=1}^{3} \alpha_{ij}\hat{x}_j, \quad x_i' = \sum_{j=1}^{3} \alpha_{ij}x_j \rightarrow \alpha_{ij}x_j$$

where α_{ij} is the cosine of the angle between the primed axis $\hat{x}_i'$ and the unprimed axis $\hat{x}_j$. For example, α_{12} is the cosine of the angle between $\hat{x}_1'$ and $\hat{x}_2$. α_{ij} represents a 9-component matrix called the transformation matrix. Unlike the stress tensor, it is not symmetric ($\alpha_{ij} \neq \alpha_{ji}$).

$$\alpha_{ij} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

In matix equations, the transformation law is written

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The inverse transformation law is written

$$\hat{x}_i = \alpha_{ji} \hat{x}_j'$$

Consider the following transformation of coordinates:

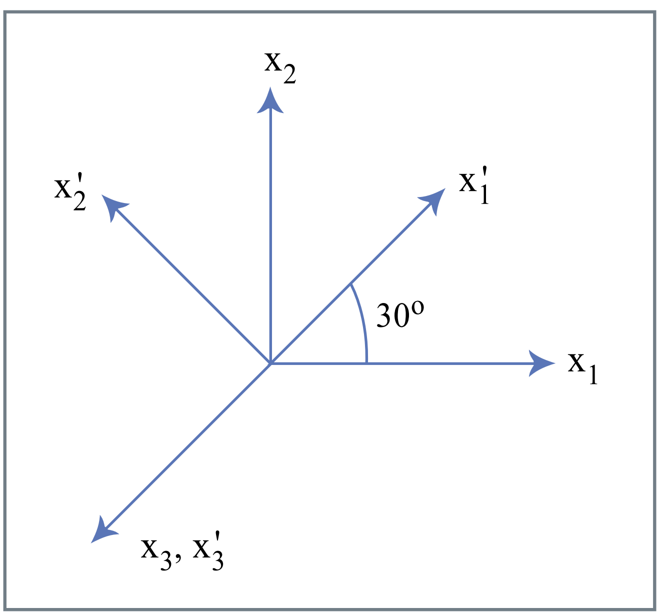


Figure 5.1a
Figure by MIT OCW.

The transformation matrix is

$$\alpha_{ij} = \begin{bmatrix} 30° & 60° & 0 \\ 120° & 30° & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The explicit transformation equations are

$$\begin{aligned} \hat{x}_1' &= \cos 30° \hat{x}_1 + \cos 60° \hat{x}_2 \\ \hat{x}_2' &= \cos 120° \hat{x}_1 + \cos 30° \hat{x}_2 \end{aligned}$$

Since $\hat{x}_i'$ and $\hat{x}_j$ are both unit length, these equations are easy to verify from the picture.

First-order tensors

First-order tensors or vectors have two components in 2D coordinates and three components in 3D coordinates. They transform according to the same laws as coordinate axes because coordinate axes are themselves vectors.

If u_j is a vector in the $\hat{x}_j$ coordinate system and u_i' is a vector in the $\hat{x}_i'$ coordinate system, then the following equations describe their transformation:

$$\begin{aligned} u_i' &= \alpha_{ij} u_j \\ u_i &= \alpha_{ji} u_j' \end{aligned}$$

Note that α_{ij} is positive if the angle is measured counterclockwise from $\hat{x}_i'$ to $\hat{x}_j$. It is negative if the angle is measured clockwise.

Second-order tensors

The transformation law for second-order tensors like stress and strain is more complicated than the transformation law for first-order tensors. It may be derived as follows:

1. Begin with the vector transformation of traction T_i to T_k':

$$T_k' = \alpha_{ki} T_i$$

2. Rewrite T_k' and T_i using Cauchy's formulas:

$$T_k' = \sigma'_{kl} n_l' \text{ and } T_i = \sigma_{ij} n_j$$

   Substitute Cauchy's formulas into the original transformation equation:

$$\sigma'_{kl} n_l' = \alpha_{ki} \sigma_{ij} n_j$$

3. Transform the normal vector n_j to n_l' and substitute into the previous equation:

$$n_j = \alpha_{lj} n_l'$$
$$\sigma'_{kl} n_l' = \alpha_{ki} \sigma_{ij} \alpha_{lj} n_l'$$

4. Cancel the n_l' term on each side and group the αs:

$$\sigma'_{kl} = \alpha_{ki} \alpha_{jl} \sigma_{ij}$$

Note that changing the position of the last α term changes the order of its subscripts.

In vector notation, the equation is

$$\underset{\approx}{\sigma}' = \alpha \underset{\approx}{\sigma} \alpha^T$$

where the double under ~ denote second-rank tensors and the superscript *T* denotes the transpose of matrix α.

Mohr's Circle

We explained before that an object resting on a slope will slide down when the shear traction on the slope is greater than or equal to the product of the normal traction and the coefficient of friction.

$$\tau = f_s \sigma_n$$

On a shallow slope, σ_n is large and the object will not slide. On a steep slow, τ is large and the object will slide. For any plane with normal $\hat{n}$, we can calculate if the plane will fail if the stress tensor σ_{ij} at the interface between the object and the slope is known. Calculate σ_n and , τ as follows:

Vector and Tensor Notation	Summation Notation
$\vec{T} = \underset{\approx}{\sigma}\hat{n}$	$\mathrm{T_i = \sigma_{ij} n_j}$
$\sigma_n = \vec{T} \cdot \hat{n}$	$\mathrm{\sigma_n = T_i n_i}$
$\tau = \vec{T} - \sigma_n \hat{n}$	$\mathrm{\tau = T_i - T_k n_k n_i}$

This method is straightforward but cumbersome. A different approach involves rotating the coordinate system such that x_1' is along $\hat{n}$. In this case σ_n and τ are much easier to derive:

$$\sigma_n = \sigma'_{11}$$
$$\tau = \sigma'_{12}$$

Mohr's circle may be derived in two or three dimensions. This lecture explains the derivation in two dimensions because it is more straightforward and the results are easier to graph and understand. The derivation assumes that x_1, x_2, and x_3 are principle directions.

Consider the following figure in which the x_i coordinate system is rotated clockwise about the x_3 axis to x_i':

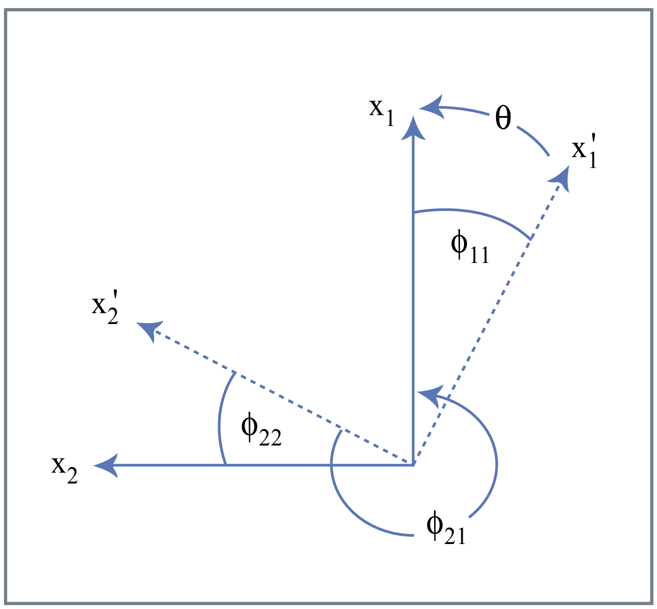


Figure 5.2
Figure by MIT OCW.

The rotation matrix is:

$$\alpha_{ij} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The stress tensor $\underset{\approx}{\sigma}$ in the x_i coordinate system is transformed to $\underset{\approx}{\sigma}'$ in the x_i' coordinate system by the following equation:

$$\underset{\approx}{\sigma}' = \alpha \underset{\approx}{\sigma} \alpha^T$$

$$\underset{\approx}{\sigma}' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11}\cos\theta & -\sigma_{22}\sin\theta & 0 \\ \sigma_{11}\sin\theta & \sigma_{22}\cos\theta & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta & (\sigma_{11} - \sigma_{22})\sin\theta\cos\theta & 0 \\ (\sigma_{11} - \sigma_{22})\sin\theta\cos\theta & \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

Use the double-angle identities for sine and cosine to simplify the expressions for the normal stress σ_{11}' and the shear stress σ_{12}' become in the new coordinate system:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\sigma_{11}' = \sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2}\cos 2\theta$$
$$\sigma_{12}' = \tau = \frac{\sigma_{11} - \sigma_{22}}{2}\sin 2\theta$$

The expression for the normal stress and shear stress can be shown graphically in shear space:

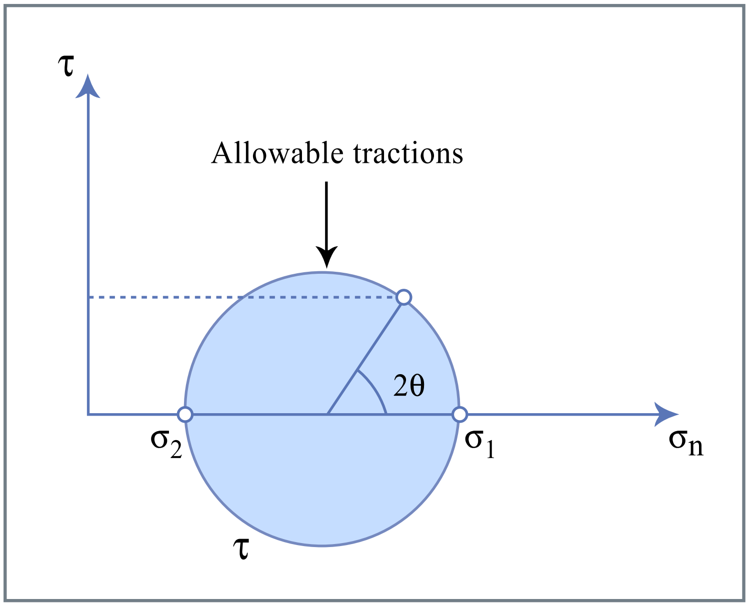


Figure 5.3
Figure by MIT OCW.

This figure is called Mohr's circle.

Mohr's circle plots in stress space. Admonton's law may also be plotted in stress space as a line with slope f_s. When this line and Mohr's circle intersect, the criterion for failure across a plane is met.

A note on signs.
As derived, assumed $\sigma_1 > \sigma_2$. Simplest case: consider $\theta = 45^\circ$

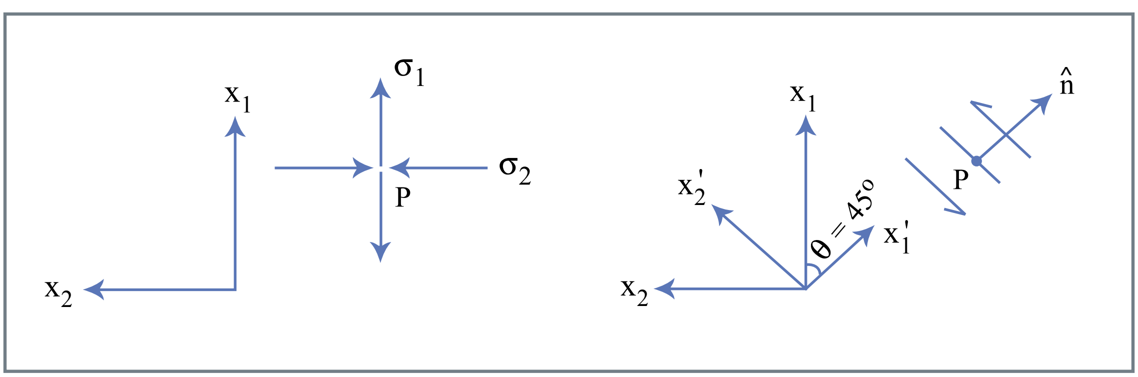


Figure 5.4
Figure by MIT OCW.

Consider several cases:

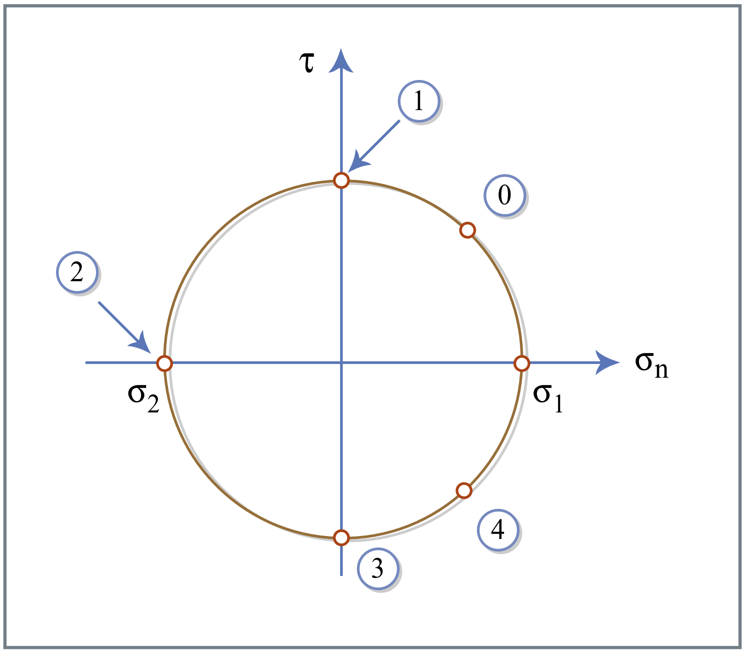


Figure 5.5

Figure by MIT OCW.

0) $2\theta = 60^{\circ}$

1) $2\theta = 90^{\circ}$

2) $2\theta = 180^{\circ}$

3) $2\theta = 270^{\circ}$

4) $2\theta = 300^{\circ}$

Back to the landslide:

Consider a common experiment in soil mechanics or rock mechanics in which scientists apply a uniaxial stress σ_2 to a cylindrical sample confined by a uniform stress σ_1.

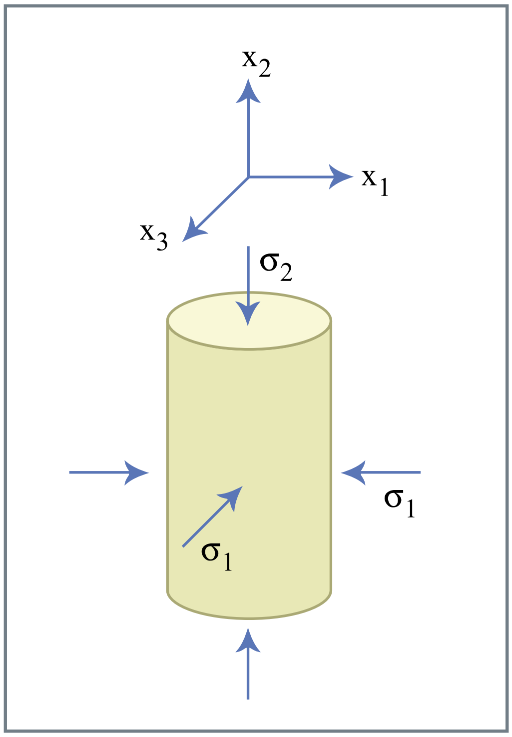


Figure 5.6

Figure by MIT OCW.

Continue increasing σ_2 until failure. $\hat{n}$ of failure plane typically at θ ; 30° from σ_1.

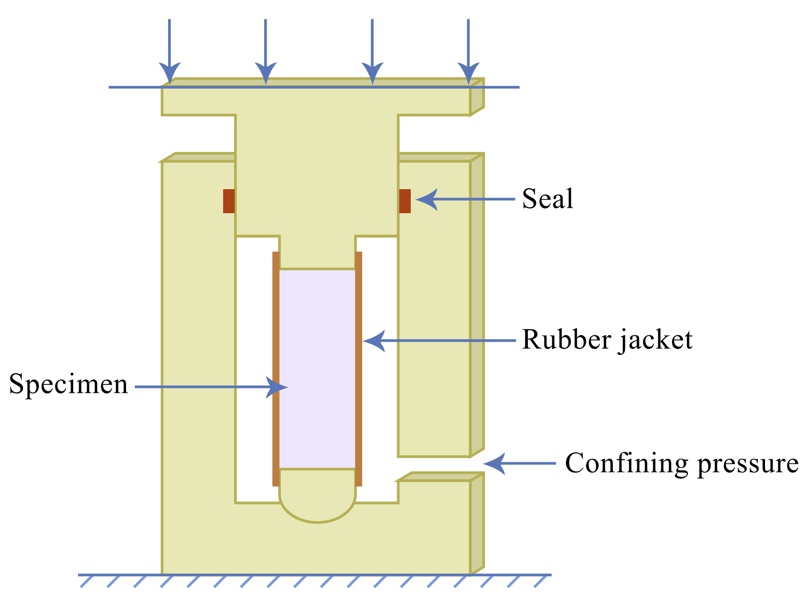


Figure 5.7. Triaxial test deformation apparatus

Figure 5.7

Figure by MIT OCW.

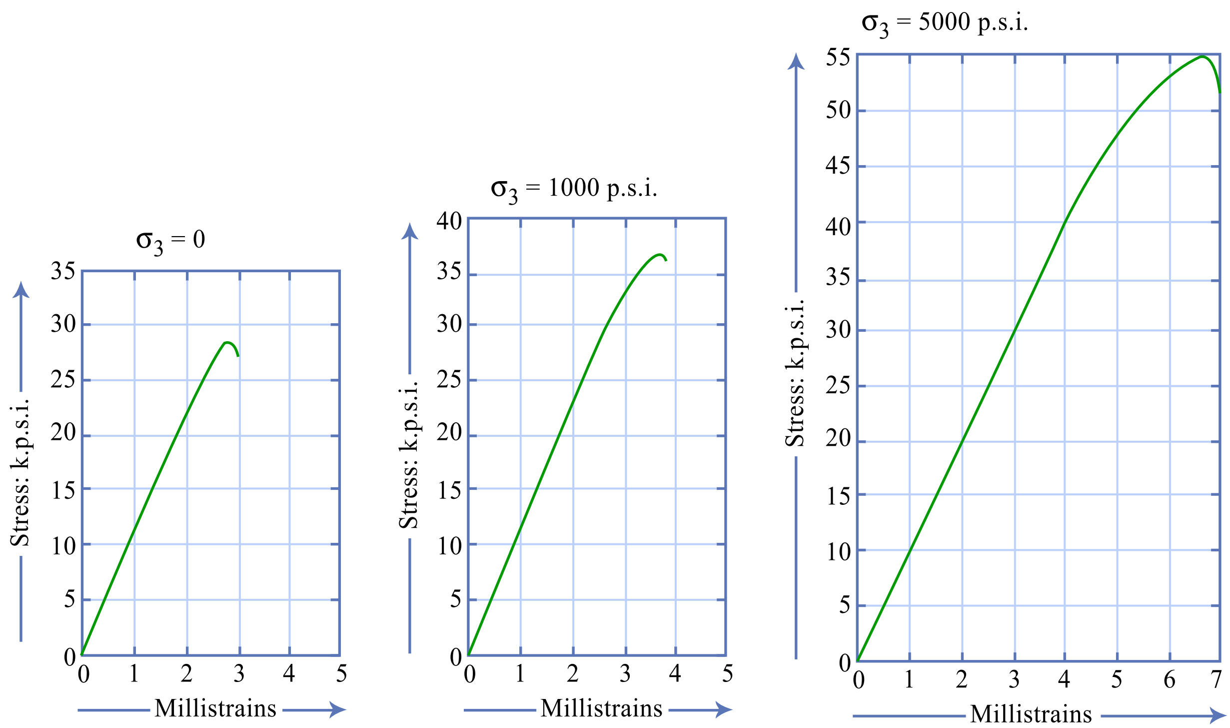
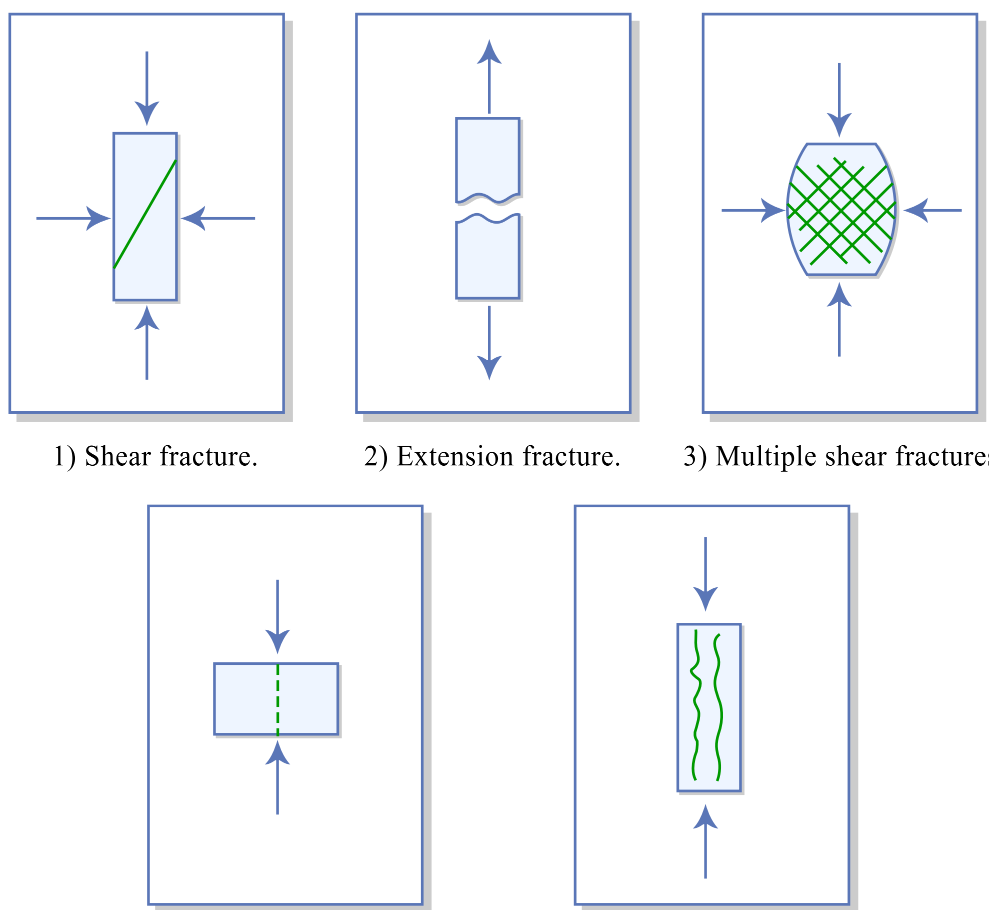


Figure 5.8. Stress-strain curves for Rand quartzite at various confining pressures.

Figure 5.8



1) Shear fracture. 2) Extension fracture. 3) Multiple shear fractures.

4) Extension fracture produced by line loads. 5) Longitudinal splitting in uniaxial compression.

Figure 5.9

Figures by MIT OCW.

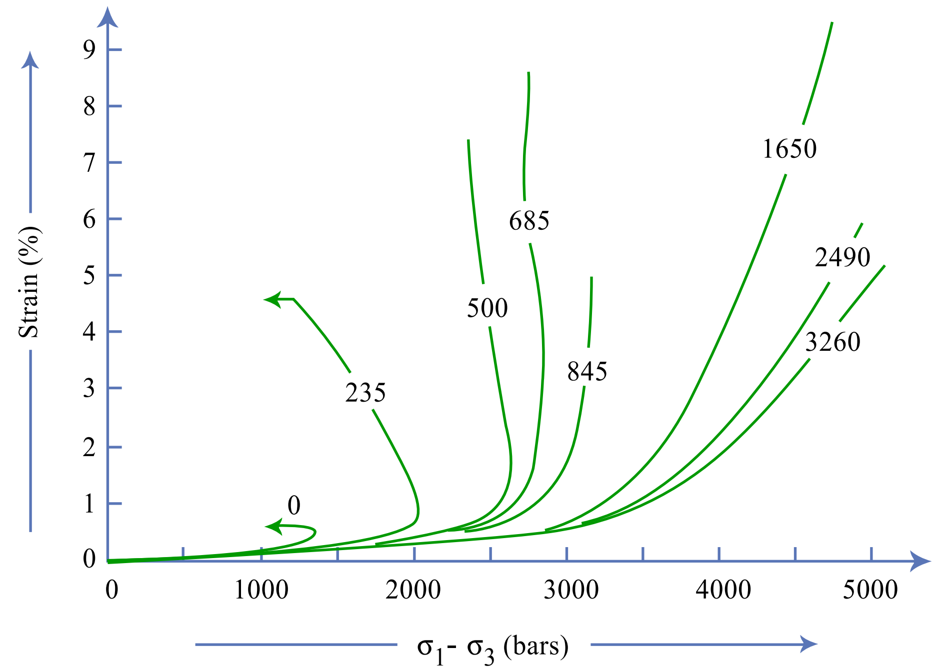


Figure 5.10. Stress-strain curves for Carrara marble at various confining pressures. The numbers on the curves are confining pressures in bars.

Figure 5.10

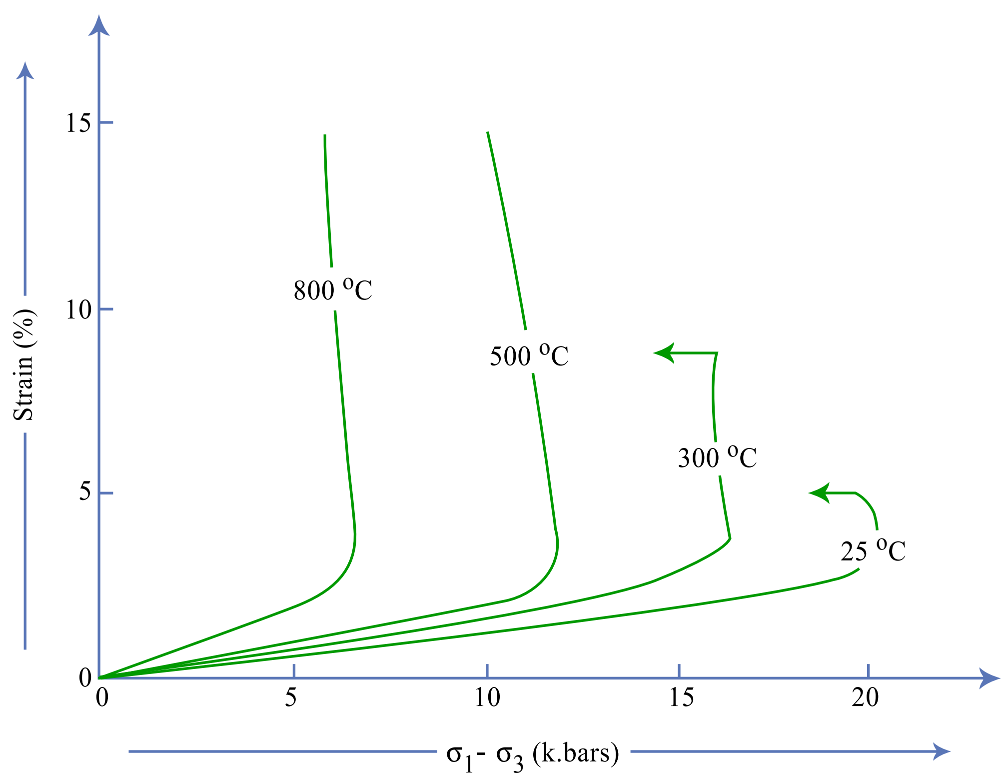


Figure 5.11. Stress-strain curves for granite at a confining pressure of 5 kilobars and various temperatures.

Figure 5.11

Figures by MIT OCW.

Analyze this using a Mohr circle diagram:

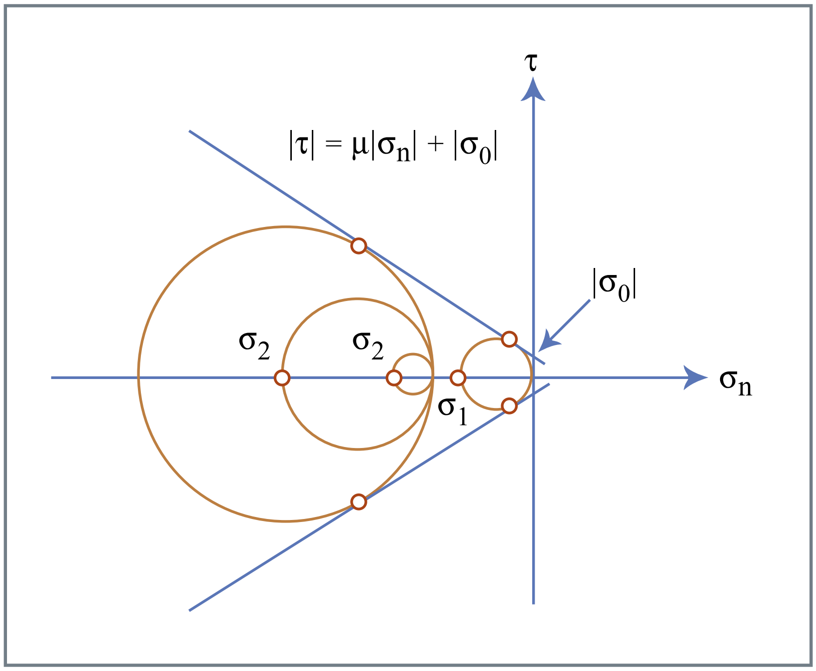


Figure 5.12

Figure by MIT OCW.

Mohr circle tangent to failure curve $\Rightarrow$ sample breaks.

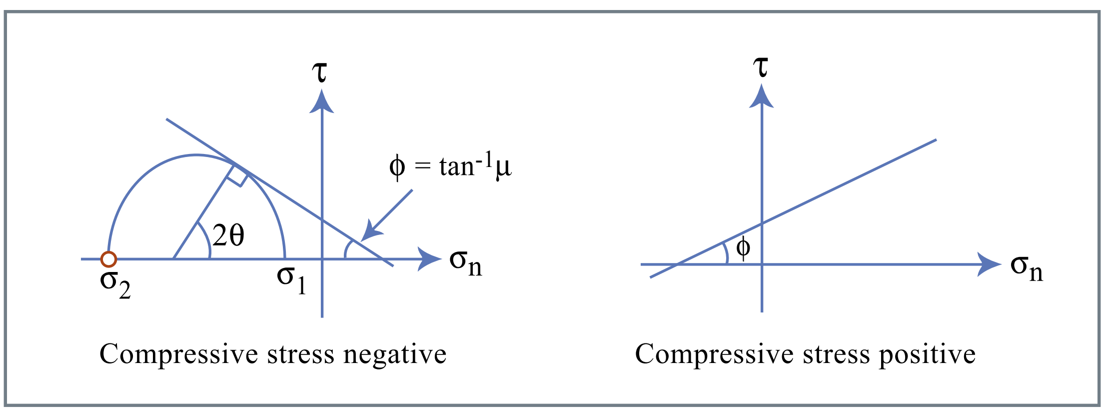


Figure 5.13

Figure by MIT OCW.

μ is the coefficient of friction. $\phi = \tan^{-1}\mu$; $2\theta = 90^{\circ} - \phi$

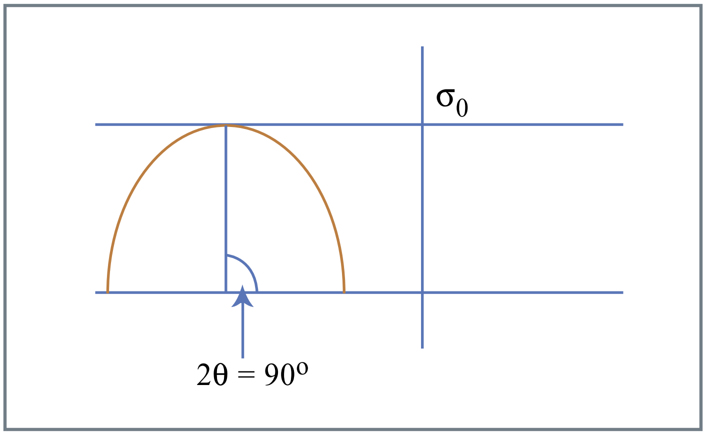


Figure 5.14

Figure by MIT OCW.

μ	θ
0.0	45°
0.6	30°
1.0	23°
$\lim\limits_{\mu\to\infty}$	0°

Since most rocks have a coefficient of friction of about 0.6, the normal vector to the failure plane is typically 30° from the direction of the least compressive stress. Another way of saying this is that the failure plane is 30° from the direction of the most compressive stress.

Styles of Faulting

Faults are large-scale failure planes. Since the normals to failure planes are in the plane containing the least compressive and the most compressive stresses and are typically at 30° from the direction of the least compressive stress, different styles of faulting can be used to infer the directions of σ_1, σ_2, and σ_3.

The following diagram shows the deviatoric stresses associated with thrust faults, normal faults, and strike-slip faults.

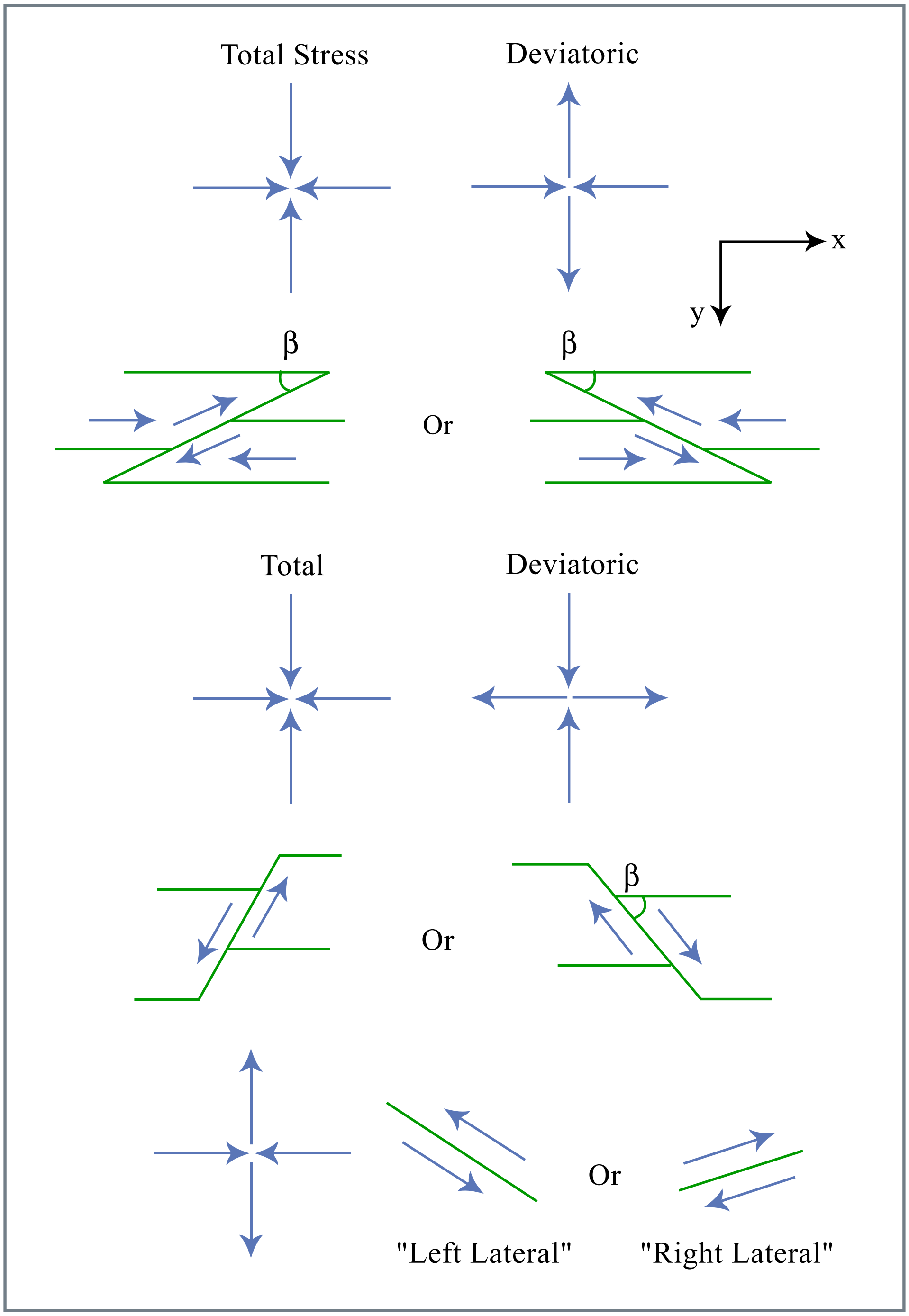


Figure 5.15

Figure by MIT OCW.