Lecture notes for 12.006J/18.353J/2.050J, Nonlinear Dynamics: Chaos

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1 Poincaré sections

The dynamical systems we study are of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = F(\vec{x}, t)$$

Systems of such equations describe a *flow* in phase space.

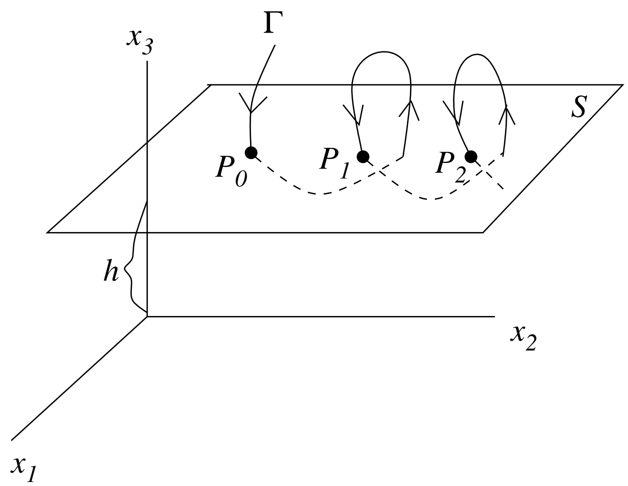
The solution is often studied by considering the trajectories of such flows.

But the phase trajectory is itself often difficult to determine, if for no other reason than that the dimensionality of the phase space is too large.

Thus we seek a geometric depiction of the trajectories in a lower-dimensional space—in essence, a view of phase space without *all* the detail.

1.1 Construction of Poincaré sections

Suppose we have a 3-D flow Γ. Instead of directly studying the flow in 3-D, consider, e.g., its intersection with a plane ($x_3 = h$):



- Points of intersection correspond (in this case) to $\dot{x}_3 < 0$ on Γ.
- Height h of plane S is chosen so that Γ continually crosses S.
- The points P_0, P_1, P_2 form the 2-D *Poincaré section.*

The Poincaré section is a continuous mapping T of the plane S onto itself:

$$P_{k+1} = T(P_k) = T\left[T(P_{k-1})\right] = T^2(P_{k-1}) = \ldots$$

Since the flow is deterministic, P_0 determines P_1, P_1 determines P_2, etc.

The Poincaré section reduces a *continuous* flow to a *discrete-time mapping.* However the time interval from point to point is not necessarily constant.

We expect some geometric properties of the flow and the Poincaré section to be the same:

- Dissipation $\Rightarrow$ areas in the Poincaré section should contract.
- If the flow has an attractor, we should see it in the Poincaré section.

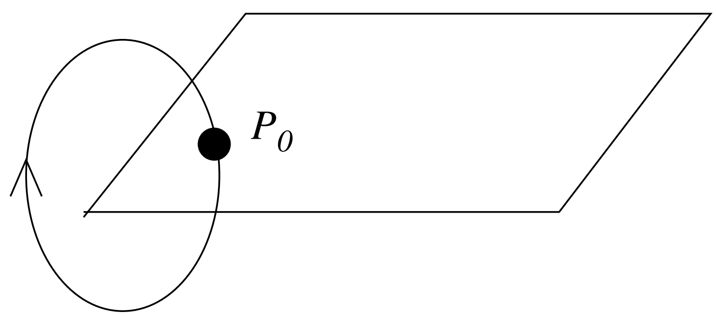
Essentially the Poincaré section provides a means to visualize an otherwise messy, possibly aperiodic, attractor.

1.2 Types of Poincaré sections

As we did with power spectra, we classify three types of flows: periodic, quasiperiodic, and aperiodic.

1.2.1 Periodic

The flow is a closed orbit (e.g., a limit cycle):



P_0 is a fixed point of the Poincaré map:

$$P_0 = T(P_0) = T^2(P_0) = \dots .$$

We proceed to analyze the stability of the fixed point.

To first order, a Poincaré map T can be described by a matrix M defined in the neighborhood of P_0:

$$M_{ij} = \left. \frac{\partial T_i}{\partial x_j} \right|_{P_0} .$$

In this context, M is called a *Floquet matrix.* It describes how a point $P_0 + \delta$ moves after one intersection of the Poincaré map.

A Taylor expansion about the fixed point yields:

$$T_i(P_0+\delta) \;\simeq\; T_i(P_0) \;+\; \left.\frac{\partial T_i}{\partial x_1}\right|_{P_0} \cdot \delta_1 \;+\; \left.\frac{\partial T_i}{\partial x_2}\right|_{P_0} \cdot \delta_2, \qquad i=1,2$$

Since $T(P_0) = P_0$,

$$T(P_0+\delta) \simeq P_0 + M\delta$$

Therefore

$$\begin{aligned} T\Big(T(P_0+\delta)\Big) &\simeq T(P_0+M\delta) \\ &\simeq T(P_0) + M^2\delta \\ &\simeq P_0 + M^2\delta \end{aligned}$$

After m interations of the map,

$$T^m(P_0+\delta) - P_0 \simeq M^m\delta.$$

Stability therefore depends on the properties of M.

Assume that δ is an eigenvector of M. (There will always be a projection onto an eigenvector.) Then

$$M^m\delta = \lambda^m\delta,$$

where λ is the corresponding eigenvalue.

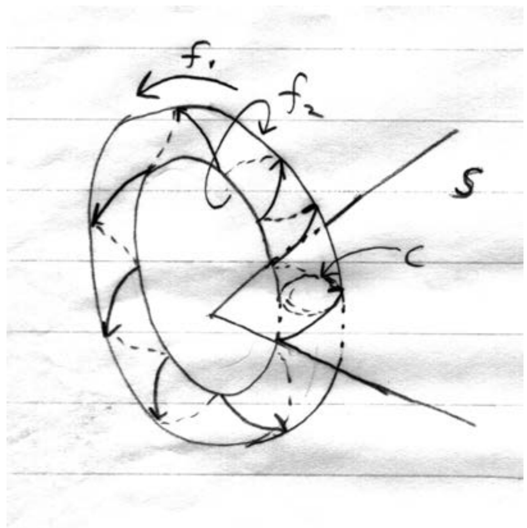
Therefore

$$\begin{aligned} |\lambda| \;&<\; 1 \;\Rightarrow\; \text{linearly stable} \\ |\lambda| \;&>\; 1 \;\Rightarrow\; \text{linearly unstable} \end{aligned}$$

Conclusion: a periodic map is unstable if one of the eigenvalues of the Floquet matrix crosses the unit circle in the complex plane.

1.2.2 Quasiperiodic flows

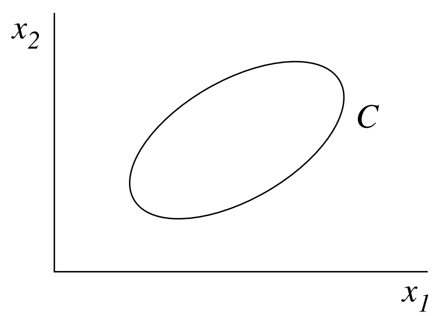
Consider a 3-D flow with two fundamental frequencies f_1 and f_2. The flow is a torus T^2:



The points of intersection of the flow with the plane S appear on a closed curve C.

As with power spectra, the form of the resulting Poincaré section depends on the ratio f_1/f_2:

- *Irrational* f_1/f_2. The frequencies are called *incommensurate.* The closed curve C appears continuous, e.g.



  - The trajectory on the torus T^2 never repeats itself exactly.
  - The curve is not traversed continuously, but rather

$$T(C) = \text{ finite shift along } C.$$

- *Rational* f_1/f_2.
  - f_1 and f_2 are *frequency locked.*
  - There are finite number of intersections (points) along the curve C.

– Trajectory repeats itself after n_1 revolutions and n_2 rotations.

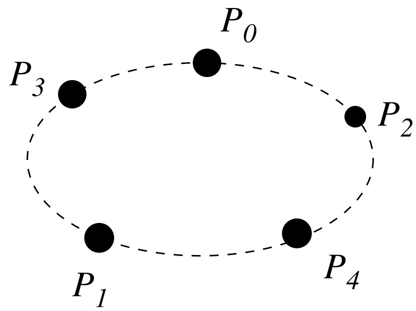
– The Poincaré section is periodic with

$$\text{period } = n_1/f_1 = n_2/f_2$$

– The Poincaré section contains just n_1 points. Thus

$$P_i = T^{n_1}(P_i)$$

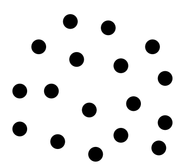
– Example, $n_1 = 5$:



1.2.3 Aperiodic flows

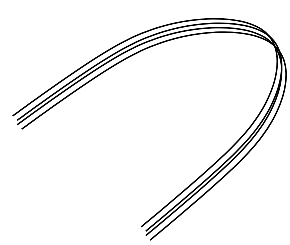
Aperiodic flows may no longer lie on some reasonably simple curve.

In an extreme case, one has just a point cloud:



This would be expected for statistical white noise.

Deterministic aperiodic systems often display more order, however. In some cases they create mild departures from a simple curve, e.g.



Such cases arise from strong dissipation (and the resulting contraction of areas in phase space).

It then becomes useful to define a coordinate x that falls roughly along this curve, and to study the iterates of x. This is called a *first return map.*

1.3 First-return maps

First return maps are 1-D reductions of the kind of 2-D Poincaré maps that we have been considering.

Such maps are of the form

$$x_{k+1} = f(x_k).$$

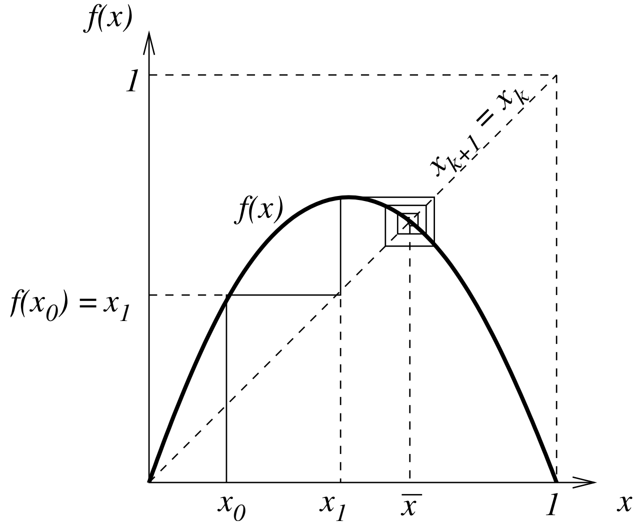
We will study these extensively at the end of the course.

We shall give particular attention to the following quadratic mapping of the unit interval onto itself:

$$x_{k+1} = 4\mu x_k(1 - x_k), \qquad 0 \leq \mu \leq 1.$$

The mapping is easily described graphically. The quadratic rises from $x = 0$, falls to $x = 1$, and has its maximum at $x = 1/2$, where it rises to height μ.

Consider, for example, the case $\mu = 0.7$:



Eventually the interations converge to $x = \bar{x}$, which is where the diagonal (the identity map $x_{k+1} = x_k$) intersects $f(x)$.

Thus $\bar{x}$ is a fixed point of f, i.e.,

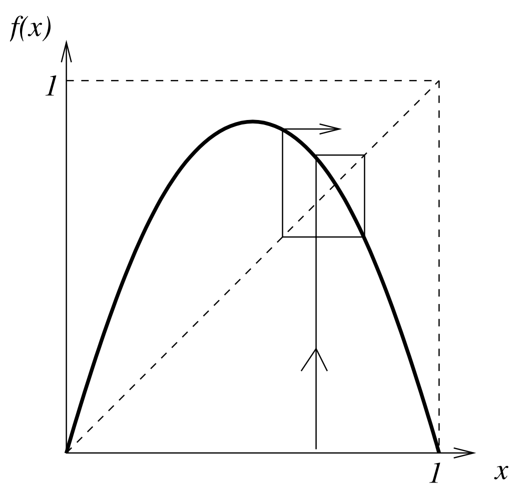
$$\bar{x} = f(\bar{x})$$

Another fixed point is $x = 0$, since $f(0) = 0$.

However we can see graphically that $x = 0$ is unstable; iterates initiated near $x = 0$ still converges to $\bar{x}$.

Thus $x = 0$ is an *unstable* fixed point, while $x = \bar{x}$ is *stable.*

What determines stability? Consider graphically the case $\mu = 0.9$:



We infer that the slope $f'(\bar{x})$ determines whether $\bar{x}$ is stable. We proceed to show this formally.

Suppose x^* is *any* fixed point such that

$$x^* = f(x^*).$$

Define

$$x_k = x^* + \varepsilon_k, \qquad \varepsilon_k \text{ small.}$$

In general, our mappings are described by

$$x_{k+1} = f(x_k).$$

Then

$$
\begin{aligned}
x^* + \varepsilon_{k+1} &= f(x^* + \varepsilon_k) \\
&= f(x^*) + f'(x^*)\varepsilon_k + O(\varepsilon_k^2)
\end{aligned}
$$

Therefore

$$\varepsilon_{k+1} \simeq f'(x^*)\varepsilon_k.$$

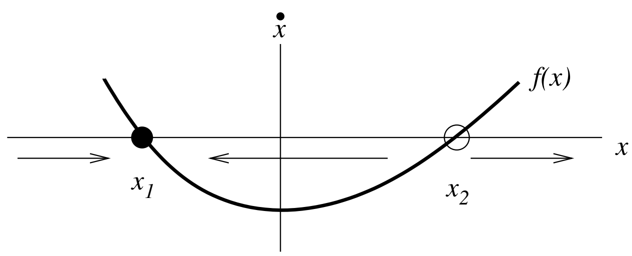
Thus

$$|f'(x^*)| < 1 \;\Rightarrow\; \text{stability.}$$

It is instructive to compare the stability of 1-D maps to the stability of the 1-D flow

$$\dot{x} = f(x).$$

Recall that the direction of flow depends on the sign of $f(x)$ and that the stability at x^* depends on the sign of $f'(x^*)$:



Whereas the stability of a continuous 1-D flow f depends on the sign of $f'(x^*)$, the stability of a 1-D map depends on the magnitude $|f'(x^*)|$.

In higher dimensions this same distinction holds for the eigenvalues λ of the Jacobian (which, in the case of mappings, we have called the Floquet matrix). That is, the sign of $\mathrm{Re}(\lambda)$ determines the stability of flows, whereas the magnitude $|\lambda|$ is the relevant quantity for maps.

1.4 Relation of flows to maps

We now consider explicitly how flows may be related to maps.

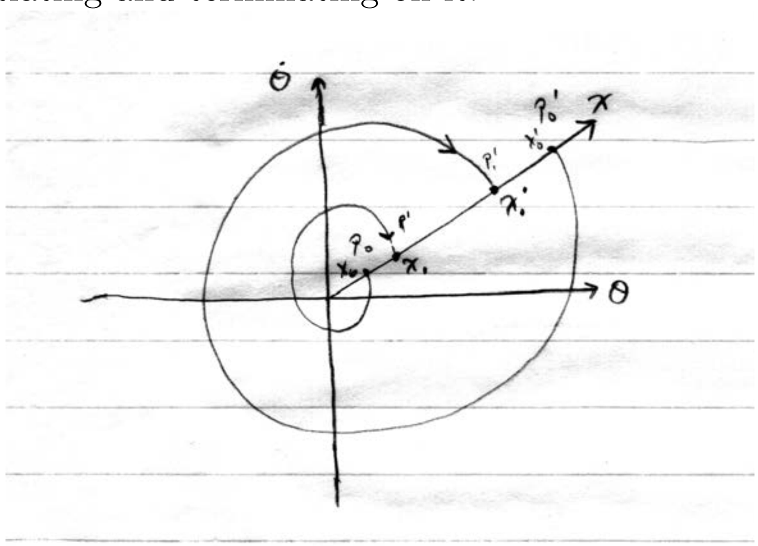
1.4.1 Example 1: the van der Pol equation

Consider again the van der Pol equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \varepsilon(\theta^2 - 1)\frac{\mathrm{d}\theta}{\mathrm{d}t} + \theta = 0$$

Recall that for $\varepsilon > 0$ the rest position is unstable and that the system has a limit cycle.

We draw a ray emanating from the origin, and consider two representative trajectories initiating and terminating on it:



Let x_k be the position of the kth intersection of the trajectory with the ray. There is then some mapping f such that

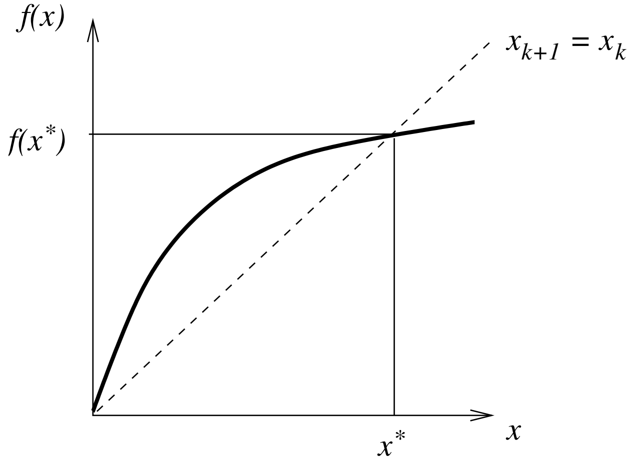
$$x_{k+1} = f(x_k).$$

The precise form of $f(x)$ is unknown, but physical and mathematical reasoning allows us to state some of its properties:

- f maps x_k to a unique x_{k+1}.
- f is continuous.
- $f'(x) > 1$ near the origin (divergent spirals).
- $f'(x) < 1$ far from the origin (convergent spirals).

- $f'(x) > 0$ for all $x > 0$ (since $f(x + \delta) > f(x)$).

The simplest form of f is therefore a curve rising steeply from the origin, followed by a gentle upward slope:



By continuity, there must be a stable fixed point x^* characterized by

$$x^* = f(x^*) \qquad \text{and} \qquad f'(x^*) < 1.$$

Thus x^* gives the effective radius of the stable limit cycle.

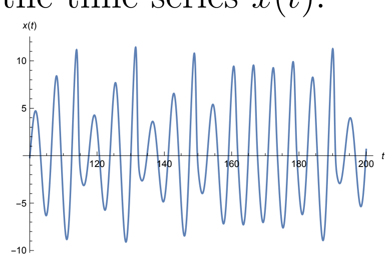
1.4.2 Example 2: Rössler attractor

Consider the following 3-D flow (the Rössler attractor):

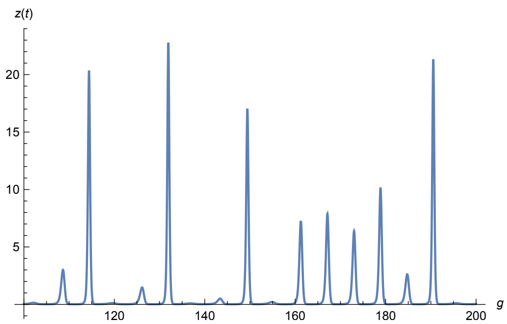
$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned}$$

a, b, and c are fixed parameters.

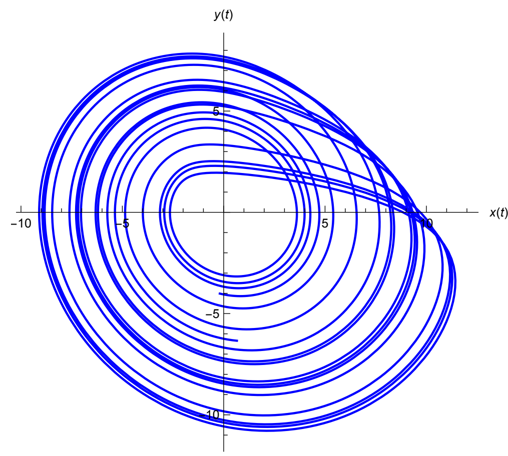
Numerical solutions yield the time series $x(t)$:



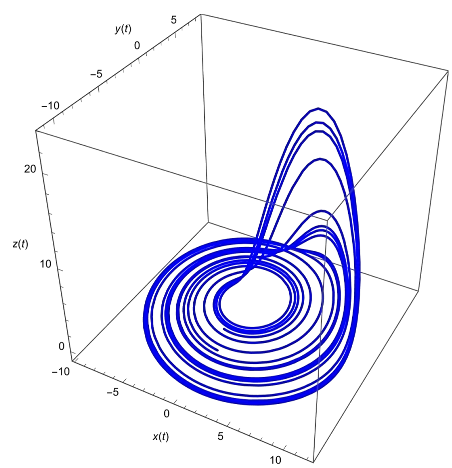
The time series $z(t)$:



The true x-y phase plane:



The true flow in 3D:



The time series—especially $z(t)$—display significant irregularity, but the 2D phase plane and the 3D flow flow display some order.

Consider now a Poincaré section in the plane

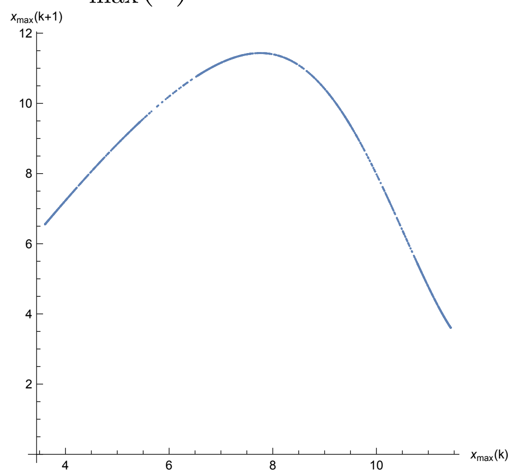
$$y + z = 0.$$

From the Rössler equations, we identify this plane with extrema in the time series $x(t)$, i.e., each intersection of the plane corresponds to

$$\dot{x} = 0.$$

Consider a sequence $x_{\max}(k)$ of such extrema, but only when the extremum is a maximum of $x(t)$.

Then plot $x_{\max}(k+1)$ vs. $x_{\max}(k)$:



Conclusions:

- The 1-D map—a Poincaré section in the plane $y = -z$–reveals that the flow contains much order.
- The time series, however, displays no apparent regularity.

This is the essence of deterministic chaos.

We proceed to show how such Poincaré sections and 1-D maps can be constructed from experimental data.

1.4.3 Example 3: Reconstruction of phase space from experimental data

Suppose we measure some signal $x(t)$ (e.g., the weather, the stock market, etc.)

In most cases it is unlikely that we can specify the equations of motion of the dynamical system that is generating $x(t)$.

How, then, may we visualize the system's phase space and its attractor?

The (heuristic but highly successful) idea is to measure any 3 *independent* quantities from $x(t)$.

For example:

- $x(t)$, $x(t+\tau)$, $x(t+2\tau)$; τ large enough for "independence," i.e., beyond an autocorrelation time. This is the most popular; it is known as the *method of delays.*
- $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$ (where the derivatives are finite differences $x_k - x_{k-1}$, etc.).

Such a representation of the attractor is not identical to the "real" phase space, but it should retain similar geometric properties.

Although we have discussed only qualitative, geometric properties. we shall see that the various representations also yield similar quantitative properties (e.g., measures of Lyaponov exponents).

You'll investigate these ideas further in the next problem set.

References

1. Bergé, P., Pomeau, Y. & Vidal, C. *Order within Chaos: Towards a Deterministic Approach to Turbulence* (John Wiley and Sons, New York, 1984).
2. Packard, N. H., Crutchfield, J. P., Farmer, J. D. & Shaw, R. S. Geometry from a time series. *Physical review letters* **45,** 712 (1980).
3. Strogatz, S. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (CRC Press, 2018).