Lecture notes for 12.006J/18.353J/2.050J, Nonlinear Dynamics: Chaos

D. H. Rothman, MIT
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1 Hénon attractor

References: [1–3]

The chaotic phenomena of the Lorenz equations may be exhibited by even simpler systems.

We now consider a discrete-time, 2-D mapping of the plane into itself. The points in $\mathbb{R}^2$ are considered to be the the Poincaré section of a flow in higher dimensions, say, $\mathbb{R}^3$.

The restriction that $d > 2$ for a strange attractor does not apply, because maps generate discrete points; thus the flow is not restricted by continuity (i.e., lines of points need not be parallel).

1.1 The Hénon map

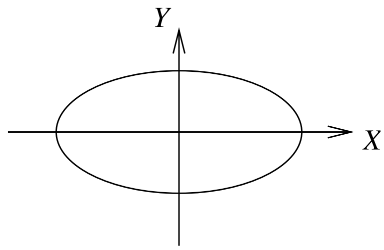
The discrete time, 2-D mapping of Hénon is

$$X_{k+1} = Y_k + 1 - \alpha X_k^2$$

$$Y_{k+1} = \beta X_k$$

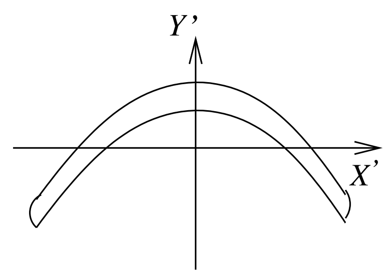
- α controls the nonlinearity.
- β controls the dissipation.

Pictorially, we may consider a set of initial conditions given by an ellipse:



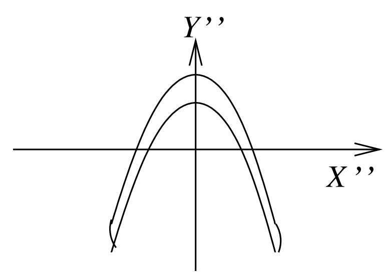
Now bend the elllipse, but preserve the area inside it (we shall soon quantify area preservation):

Map T_1 :
$$
\begin{aligned}
X' &= X \\
Y' &= 1 - \alpha X^2 + Y
\end{aligned}
$$



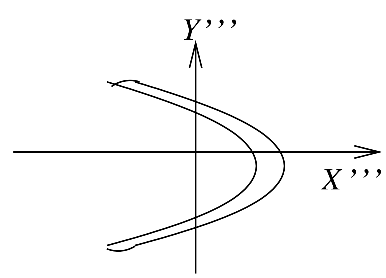
Next, contract in the x-direction ($|\beta| < 1$)

Map T_2 :
$$
\begin{aligned}
X'' &= \beta X' \\
Y'' &= Y'
\end{aligned}
$$



Finally, reorient along the x axis (i.e. flip across the diagonal).

Map T_3 :
$$
\begin{aligned}
X''' &= Y'' \\
Y''' &= X''
\end{aligned}
$$



The composite of these maps is

$$T = T_3 \circ T_2 \circ T_1.$$

We readily find that T is the Hénon map:

$$\begin{aligned} X''' &= 1 - \alpha X^2 + Y \\ Y''' &= \beta X \end{aligned}$$

1.2 Dissipation

The rate of dissipation may be quantified directly from the mapping via the Jacobian.

We write the map as

$$\begin{aligned} X_{k+1} &= f(X_k, Y_k) \\ Y_{k+1} &= g(X_k, Y_k) \end{aligned}$$

Infinitesimal changes in mapped quantities as a function of infinitesimal changes in inputs follow

$$\mathrm{d}f = \frac{\partial f}{\partial X_k}\mathrm{d}X_k + \frac{\partial f}{\partial Y_k}\mathrm{d}Y_k$$

We may approximate, to first order, the increment ΔX_{k+1} due to small increments $(\Delta X_k, \Delta Y_k)$ as

$$\Delta X_{k+1} \simeq \frac{\partial f}{\partial X_k}\Delta X_k + \frac{\partial f}{\partial Y_k}\Delta Y_k$$

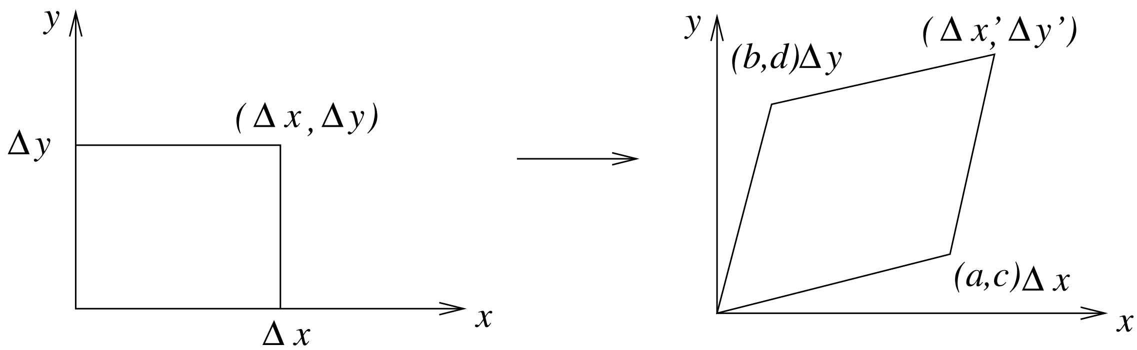
When $(\Delta X_k, \Delta Y_k)$ are perturbations about a point (x_0, y_0), we have, to first order,

$$\begin{bmatrix} \Delta X_{k+1} \\ \Delta Y_{k+1} \end{bmatrix} = \begin{bmatrix} f'_{X_k}(x_0, y_0) & f'_{Y_k}(x_0, y_0) \\ g'_{X_k}(x_0, y_0) & g'_{Y_k}(x_0, y_0) \end{bmatrix} \begin{bmatrix} \Delta X_k \\ \Delta Y_k \end{bmatrix}.$$

Rewrite simply as

$$\begin{bmatrix} \Delta x' \\ \Delta y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

Geometrically, this system describes the transformation of a rectangular area determined by the vertex $(\Delta x, \Delta y)$ to a parallelogram as follows:



Here we have taken account of transformations like

$$\begin{aligned} (\Delta x, 0) &\rightarrow (a\Delta x, c\Delta x) \\ (0, \Delta y) &\rightarrow (b\Delta y, d\Delta y) \end{aligned}$$

If the original rectangle has unit area (i.e., $\Delta x \Delta y = 1$), then the area of the parallelogram is given by the magnitude of the cross product of (a, c) and (b, d), or, in general, the Jacobian determinant

$$J = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} \dfrac{\partial X_{k+1}}{\partial X_k} & \dfrac{\partial X_{k+1}}{\partial Y_k} \\ \\ \dfrac{\partial Y_{k+1}}{\partial X_k} & \dfrac{\partial Y_{k+1}}{\partial Y_k} \end{vmatrix}_{(x_0, y_0)}$$

Therefore

$$\begin{aligned} |J| > 1 &\implies \text{dilation} \\ |J| < 1 &\implies \text{contraction} \end{aligned}$$

For the Hénon map,

$$J = \begin{vmatrix} -2\alpha X_k & 1 \\ \beta & 0 \end{vmatrix} = -\beta$$

Thus areas are multipled at each iteration by $|\beta|$.

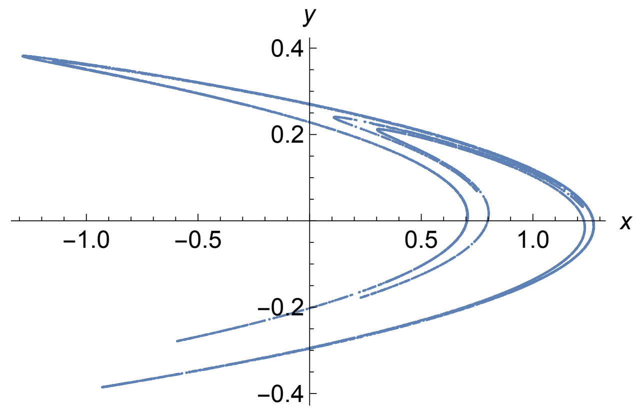
After k iterations of the map, an initial area a_0 becomes

$$a_k = a_0 |\beta|^k.$$

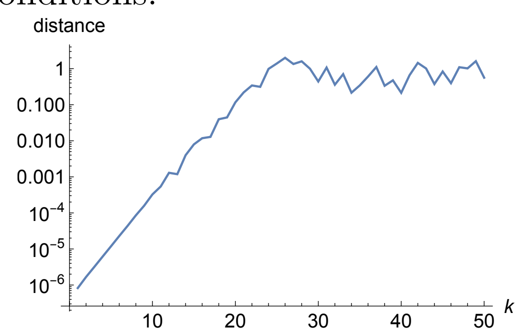
1.3 Numerical simulations

Hénon chose $\alpha = 1.4$, $\beta = 0.3$. The dissipation is thus considerably less than the factor of 10^{-6} in the Lorenz model.

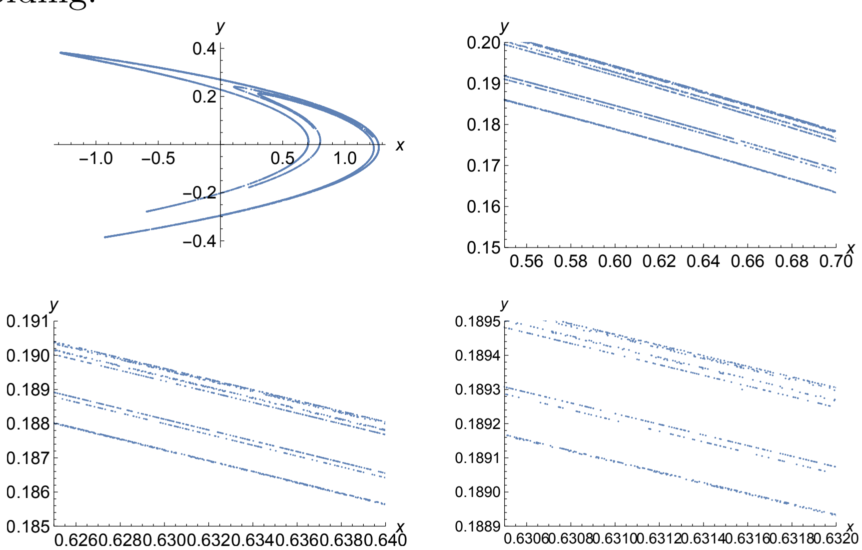
The attractor:



Sensitivity to initial conditions:



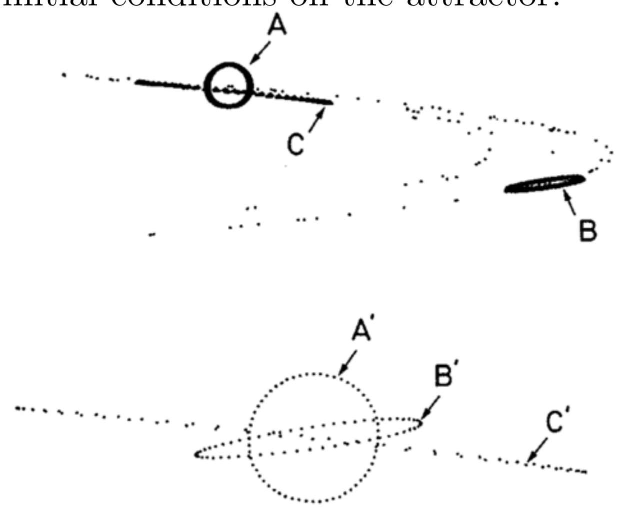
The weak dissipation allows one to see the fractal structure induced by the repetitive folding:



Note the apparent scale-invariance: at each magnification of scale, we see that the upper line is composed of 3 separate lines.

The fractal dimension $D = 1.26$. (We shall soon discuss how this is computed.)

The action of the Hénon map *near* the attractor is evident in the deformation of a small circle of initial conditions on the attractor:

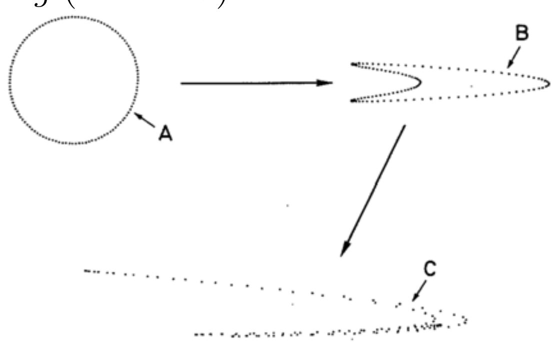


Ref. [2], Figure VI.22

The circle stretches in one dimension, by a factor Λ_1, and is compressed in the other, by a factor Λ_2. While we don't know Λ_1 and Λ_2, we do know their product: $\Lambda_1 \Lambda_2 = \beta$.

The larger of the two Λ's is related to the exponential rate at which the separation of two initial conditions grows.

At the larger scale of the attractor itself (A), we can see the combined effects of *stretching* and *folding* (B and C):



Ref. [2], Figure VI.23

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References

1. Hénon, M. A two-dimensional mapping with a strange attractor. *Commun. Math. Phys.* **50,** 69–77 (1976).
2. Bergé, P., Pomeau, Y. & Vidal, C. *Order within Chaos: Towards a Deterministic Approach to Turbulence* (John Wiley and Sons, New York, 1984).
3. Strogatz, S. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (CRC Press, 2018).

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