Lecture notes for 12.006J/18.353J/2.050J, Nonlinear Dynamics: Chaos

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Contents

1 Fractals

References: [1–4]

We now proceed to quantify the "strangeness" of strange attractors. There are two processes of interest, each associated with a measurable quantity:

- sensitivity to initial conditions, quantified by Lyaponov exponents.
- repetitive folding of attractors, quantified by the fractal dimension.

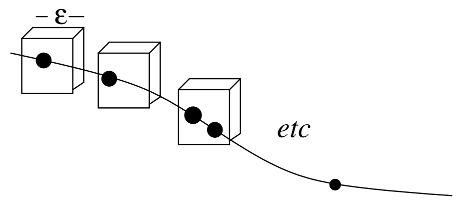
Now we consider fractals, and defer Lyaponov exponents to the next lecture.

We shall see that the fractal dimension can be associated with the effective number of degrees of freedom that are "excited" by the dynamics, e.g.,

- the number of independent variables;
- the number of oscillatory modes; or
- the number of peaks in the power spectrum

1.1 Definition

Consider an attractor A formed by a set of points in a p-dimensional space:



We contain each point within a (hyper)-cube of linear dimension ε.

Let $N(\varepsilon)$ = smallest number of cubes of size ε needed to cover A.

Then if

$$N(\varepsilon) = C\varepsilon^{-D}, \qquad \text{as } \varepsilon \to 0, \qquad C = \text{const.}$$

then D is called the *fractal* (or *Hausdorf*) dimension.

Solve for D (in the limit $\varepsilon \to 0$):

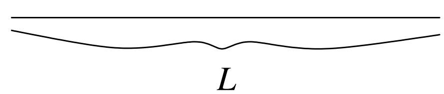
$$D = \frac{\ln N(\varepsilon) - \ln C}{\ln(1/\varepsilon)}.$$

Since $\ln C/\ln(1/\varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain the formal definition

$$D = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}.$$

1.2 Examples

Suppose A is a line segment of length L:



Then the "boxes" that cover A are just line segments of length ε, and it is obvious that

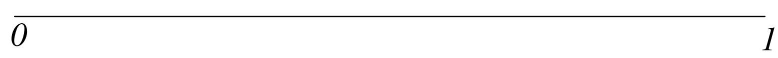
$$N(\varepsilon) = L\varepsilon^{-1} \implies D = 1.$$

Next suppose A is a surface or area S. Then

$$N(\varepsilon) = S\varepsilon^{-2} \implies D = 2.$$

But we have yet to learn anything from D.

Consider instead the Cantor set. Start with a unit line segment:



The successively remove the middle third:

Note that the structure is *scale-invariant*: from far away, you see the middle 1/3 missing; closer up, you see a different middle 1/3 missing.

The effect is visually similar to that seen in the Lorenz, Hénon, and Rössler attractors.

The fractal dimension of the Cantor set is easily calculated from the definition of D:

$$\begin{aligned} \text{Obviously,} \qquad N\left(\varepsilon = \frac{1}{3}\right) &= 2 \\ \text{Then} \qquad N\left(\varepsilon = \frac{1}{9}\right) &= 4 \\ N\left(\frac{1}{27}\right) &= 8\ldots \end{aligned}$$

Thus

$$N\left(\frac{1}{3^m}\right) = 2^m.$$

Taking $\varepsilon = 1/3$ and using the definition of D,

$$D = \lim_{m\to\infty} \frac{\ln 2^m}{\ln 3^m} = \frac{\ln 2}{\ln 3} \simeq 0.63$$

1.3 Correlation dimension ν

We proceed now to an alternative procedure for the calculation of the fractal dimension, which offers additional (physical) insight.

Rather than calculating the fractal dimension via its definition, we calculate the *correlation dimension* ν.

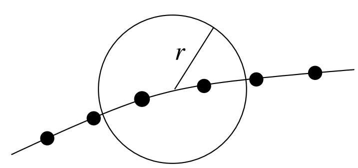
We shall show that $\nu \le D$. But first we define it.

1.3.1 Definition

Consider a set of points distributed on a plane.

Let $N(r)$ = number of points located inside a circle of radius r.

Assume the points are uniformly distributed on a curve like



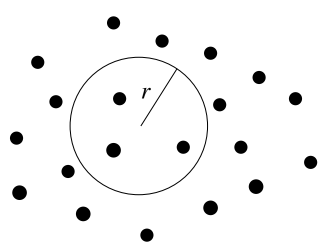
For r sufficiently small compared to the curvature of the curve, we have

$$N(r) \propto r$$

or

$$N(r) \propto r^{\nu}, \qquad \nu = 1.$$

Now assume the points are uniformly distributed along a surface in two dimensions:



Now

$$N(r) \propto r^2 \implies \nu = 2.$$

Next, reconsider the Cantor set:

We expect that $N(r)$ will grow more slowly than r.

Indeed, calculations show that $\nu \simeq 0.63 = D$, just as before.

1.3.2 Computation

Our implicit definition of ν is clearly generalized by considering

- an attractor in a p-dimensional space, and
- $N(r) =$ number of points in a p-dimensional hypersphere of radius r.

For a time series $x(t)$, we reconstruct a p-dimensional phase space with the coordinates

$$x(t), x(t+\tau), x(t+2\tau), \ldots x(t+(p-1)\tau) = \vec{x}(t).$$

Suppose there are m points on the attractor. We quantify the spatial correlation of these points by defining

$$C(r) = \lim_{m\to\infty} \frac{1}{m^2} \left(\text{number of pairs } i, j \text{ for which } |\vec{x}_i - \vec{x}_j| < r\right).$$

More formally,

$$C(r) = \lim_{m\to\infty} \frac{1}{m^2} \sum_i^m \sum_j^m H(r - |\vec{x}_i - \vec{x}_j|)$$

where

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & \text{else.} \end{cases}$$

The summation is performed by centering hyperspheres on *each* of the m points.

In practice, one *embeds* the signal $x(t)$ in a phase space of dimension p, for

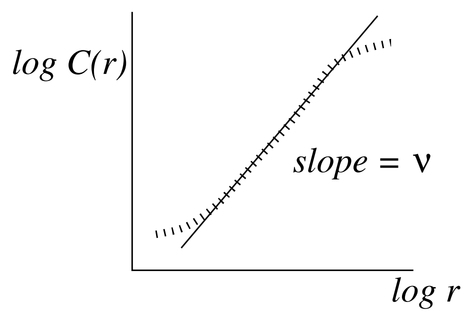
$$p = 2, 3, 4, 5, \ldots$$

p is called the *embedding dimension.*

For each p, we calculate $C(r)$. Then, assuming

$$C(r) = r^{\nu}$$

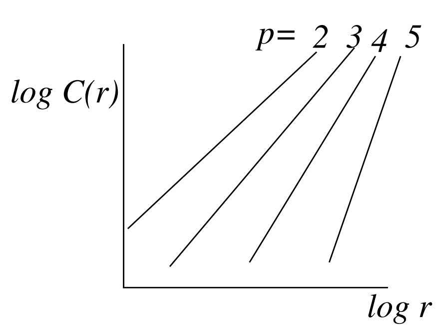
we plot $\log C$ vs. $\log r$ and estimating the slope ν:



Consider the example of white noise. Then $x(t)$ is a series of uncorrelated random numbers, and we expect

$$C(r) \propto r^p, \qquad p = \text{embedding dimension.}$$

Graphically, one expect a series of plots like



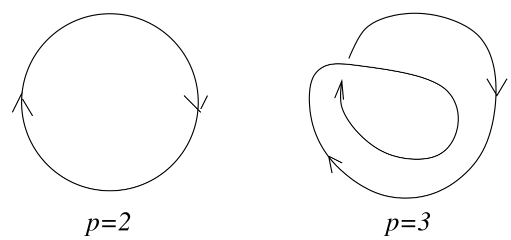
Here

$$\nu(p) = p,$$

a consequence of the fact that white noise has as many degrees of freedom (i.e., independent "modes") as there are data points.

Consider instead $X(t)$ = periodic function, i.e., a limit cycle, with only one fundamental frequency.

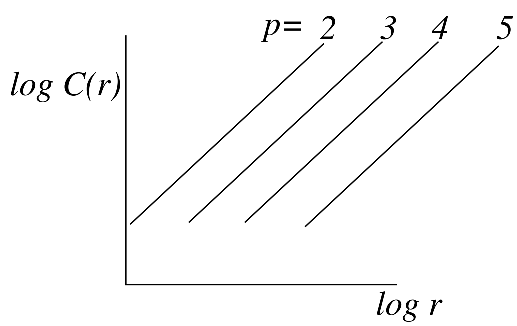
Then the attractor looks like



Provided that r is sufficiently smaller than the curvature of the limit cycle, we expect

$$C(r) \propto r^1, \qquad \text{for } p = 2, 3, 4, \ldots$$

Graphically, we obtain



and therefore

$$\nu(p) = 1, \qquad \text{independent of } p.$$

We conclude that ν measures something related to the "number of degrees of freedom" needed to parameterize an attractor.

Specifically, suppose a dynamical regime has n oscillatory modes. The attractor is then a torus T^n, and we expect

$$C(r) \propto r^n.$$

Thus

$$p \leq n \implies C(r) \propto r^p$$

and

$$p > n \implies C(r) \propto r^n, \qquad \text{independent of } p.$$

Conclusion: If, for embedding dimensions $p \geq p_0$, ν is independent of p, then ν is the number of degrees of freedom *excited* by the system.

This conclusion provides for an appealing conjecture: since white noise gives

$$\nu(p) = p,$$

ν independent of p (and reasonably small) implies that the signal is deterministic, and characterizable by $\sim\nu$ variables.

There are some practical limitations:

- r must be small compared to the attractor size.
- r and m must be large enough for reasonable statistics.
- Experimental noise, non-stationary time series, and difficulties extrapolating $r \to 0$ can also be a problem.

1.4 Relationship of ν to D

The correlation dimension is not strictly the same as the fractal dimension, however it can be. We now derive their mathematical relation.

Suppose we cover an attractor A with $N(r)$ hypercubes of size r.

If the points are *uniformly* distributed on A, the probability that a point falls into the ith hypercube is

$$p_i = 1/N(r).$$

By definition, for an attractor containing m points,

$$C(r) = \lim_{m\to\infty} \frac{1}{m^2} \sum_i^m \sum_j^m H(r - |\vec{x}_i - \vec{x}_j|), \qquad H(x) = \begin{cases} 1 & x > 0 \\ 0 & \text{else} \end{cases}$$

$C(r)$ measures the number of pairs of points within a distance r of each other. In a box of size r, there are on average mp_i points, all within the range r. Therefore, within a factor of $O(1)$ (i.e., ignoring box boundaries and factors of two arising from counting pairs twice),

$$\begin{aligned} C(r) &\simeq \frac{1}{m^2} \sum_{i=1}^{N(r)} (mp_i)^2 \\ &= \sum_{i=1}^{N(r)} p_i^2 \end{aligned}$$

Then, using angle brackets to represent mean quantities, we have, from Schwartz's inequality,

$$C(r) \;=\; N(r)\,\langle p_i^2\rangle \;\geq\; N(r)\,\langle p_i\rangle^2 \;=\; \frac{1}{N(r)}.$$

If the attractor has fractal dimension D, then

$$N(r) \propto r^{-D}, \qquad r \to 0.$$

The definition of the correlation dimension ν, on the other hand, gives

$$C(r) \propto r^{\nu}.$$

Substituting these relations into both sides of the inequality, we find

$$r^{\nu} \geq r^{D}$$

Thus as $r \to 0$, we see that

$$\nu \leq D$$

The equality is obtained when $\langle p_i^2\rangle = \langle p_i\rangle^2$.

Thus $\nu < D$ results from non-uniformity of points on the attractor.

References

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2. Bergé, P., Pomeau, Y. & Vidal, C. *Order within Chaos: Towards a Deterministic Approach to Turbulence* (John Wiley and Sons, New York, 1984).
3. Grassberger, P. Grassberger-Procaccia algorithm. *Scholarpedia* **2.** revision #91330, 3043 (2007).
4. Strogatz, S. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (CRC Press, 2018).