Lecture notes for 12.006J/18.353J/2.050J, Nonlinear Dynamics: Chaos

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Contents

1 Introduction

This course is
12.006J/18.353J/2.050J Nonlinear Dynamics: Chaos

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1.1 Who am I?

A professor of geophysics. My current interests focus on

- How the carbon cycle works, including its relation to abrupt climate change (via "tipping points")

- Dynamical mechanisms underlying the coevolution of life and the environment, and catastrophes such as mass extinctions.
- Complex systems in general.

I created this course long ago, but have not taught it since 2006. I'm delighted to teach it again, and am exploring many ways of reinvigorating its content.

1.2 What is this course?

An *undergraduate* introduction to the theory and phenomenology of *dissipative nonlinear dynamical systems*.

Let's parse that out:

- *Dynamical system*: anything (physical, chemical, biological) that evolves with time. Here we consider systems parameterized by only a few variables (e.g., position and momentum...).
- *Dissipative*: system has some friction (e.g., viscosity). As $t \to \infty$, system approaches an *attractor* which does not depend (usually) on initial conditions (e.g., rest state of, say, a pendulum; terminal velocity of a falling object).

  Almost all systems in Nature are dissipative. Counter-examples: Solar system dynamics are *conservative* ("Hamiltonian"). Also molecular dynamics of an ideal gas (elastic collisions).
- *Nonlinear.* Nonlinear science is literally the study of systems (theoretical or real) that are *not* linear.

Let's look at the last two points more closely.

1.2.1 Nonlinear systems

The Polish-American scientist Stanislas Ulam once famously remarked that defining nonlinear science as above is "like defining the bulk of zoology by calling it the study of non-elephant animals [1]."

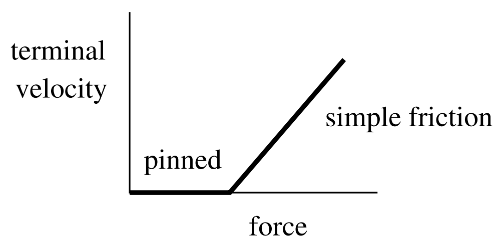
Consider, for example, the usual assumptions that

- stress $\propto$ strain;
- flux $\propto$ force; or
- current $\propto$ voltage

We often think that, e.g. pushing something twice as hard yields twice the velocity.

But consider these examples:

- Push a block with a weak force. If the force is too weak, the block sticks to surface. If the force exceeds a threshold, the block slips.

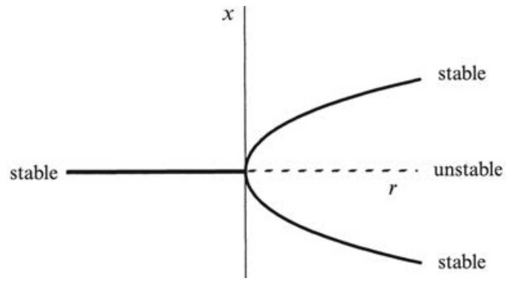
  This is the basis of "stick-slip" models for the dynamics of, e.g., earthquake faults. Or violin bows on a string.

- Make a pile of sand by adding one grain at a time. Most of the time the grains are at rest. But occasionally there are *avalanches*. Most are small, but some are quite big.

- Heat a fluid from below. If the thermal gradient is weak, heat diffuses upward but the fluid does not move

  Stronger thermal gradients: convection (fluid motion) carries warm, less dense fluid upward, and cold, more dense fluid downward.

If we place a probe somewhere in the fluid, the joint possibility of an upward or downward velocity x leads to a picture that looks like this (where r measures the thermal gradient):



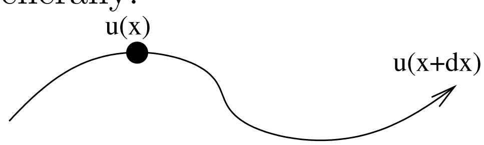
Strogatz [2], Fig. 3.4.2

Here, when r is less than a critical value r_c, the zero-velocity state is stable; above r_c it is unstable and the typical velocity grows like $(r - r_c)^{1/2}$.

We'll look at such situations more generally in the next lecture.

But for now, we note that this is an example of a general characteristic of nonlinear systems: small changes in parameters can lead to qualitatively different behavior.

- Fluid dynamics more generally:



The fluid velocity $\vec{u}(\vec{x})$ changes in part because the fluid flows; i.e.,

$$\vec{u}(\vec{x}) \to \vec{u}(\vec{x} + \mathrm{d}\vec{x})$$

But $\vec{u}$ also governs how fast this change occurs. Therefore $\mathrm{d}\vec{u}/\mathrm{d}t$ depends nonlinearly on u, and includes a change like

$$(\vec{u} \cdot \nabla)\vec{u} \sim u^2$$

i.e., a particle moves at velocity $\vec{u}$ along a velocity gradient $\nabla\vec{u}$ to a place where the velocity is different.

When this nonlinearity is weak (because u is small), flow is smooth and laminar. When it is stronger, flow becomes turbulent.

- Climate. The climate "system' involves fluids, convection, and perhaps the most nonlinear system of all: life. So it is unquestionably nonlinear.

But is it unstable? There are many known—and unknown—positive feedbacks. And plenty of examples of abrupt climate change in the past.

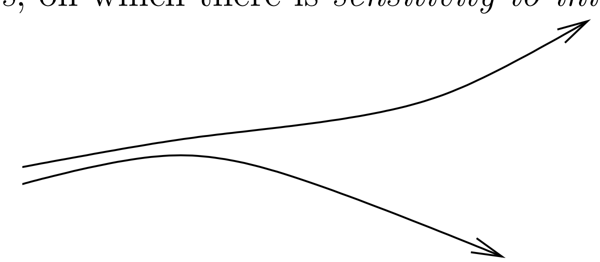
Understanding the methods and concepts in this course is a necessary first step toward determining whether climate "tipping points" exist.

- Social systems (e.g., political or economic). The same general remarks hold.

1.2.2 Dissipative systems

In dissipative systems, energy input to a system is eventually balanced by friction. The resulting "steady state," a kind of "attractor" is sometimes quite simple.

But we shall see that the combination of *nonlinearity* and *dissipation* can lead to *strange attractors*, on which there is *sensitivity to initial conditions*.



The overall idea is that small changes in initial conditions lead to large changes in the long term.

The classic example is the weather, explained first, in 1963, by MIT professor Edward Lorenz.

Lorenz's discovery was eventually termed the "butterfly effect": a butterfly that flaps its wings in, say, Brazil, can affect, at a later time (*in principle*) the weather in New York.

This *deterministic unpredictability* we call *chaos*.

The idea has now entered the cultural mainstream. But this course shows that the notion of chaos in dissipative systems is really quite non-intuitive: we'll understrand why it is possible for a system to be attracted to a statistically steady state—its attractor—regardless of initial conditions, while being sensitive to to initial conditions *on the attractor.*

1.3 Course goals

We teach:

- Elementary aspects of the theory of nonlinear dynamics and chaos.
- Phenomenology (e.g., aspects of fluid turbulence, scaling laws, experimental phenomena).
- Computer experimentation.
- Analysis of experimental data.

Our computational experiments are *exploratory.* Rather than focusing on the computation of a specific quantity (e.g., some integral), we construct simple models and compute their evolution to determine qualitative aspects of dynamics.

These qualitative dynamics are often quite general and apply to a wide array of problems in science and engineering.

Thus a major goal of the course is for students to learn why such wide-ranging applicability exists in problems that may superficially appear quite different.

1.4 Administrative details

- TA: Constantin Arnscheidt.
- All course materials will be available on Canvas.
- Prereqs: Must know o.d.e.'s (18.03). Some linear algebra (e.g., eigenvalues and eigenvectors).
- Problem sets: Some analytic, some require numerical simulation.

- We usually provide Matlab and Python codes, but modifications are often necessary. Only rudimentary coding skills are required, and they can be learned in this course.
- The objective of the numerical experiments is to impart a sense of discovery in the exploration of dynamical systems and comparison with theoretical predictions (in the spirit of Lorenz and Feigenbaum).
- Students with no experience in numerical computation may wish to consult the TA for assistance.
- Requirements
  - No exams.
  - There will be about 9 problem sets.
  - A final project: either a review of a topic in the literature, your own attempt to apply or extend what you've learned to a problem that interests you, or a combination of both. A written report will be due in the last class (Dec. 13), at which time students will also give brief presentations. You should choose your topic by Nov. 8, and submit it for approval. Further guidelines will be given.
  - Problems sets count for about 80% of the grade and the final project about 20%.
- The first pset is due next Thursday. It is easy, but we want to be sure that everyone is comfortable with the (modest) numerical computation.

1.5 Syllabus

1. Elementary nonlinear dynamics and its empirical analysis
   (a) Flows and bifurcations in 1D.
   (b) Oscillators, phase space, stability, conservation/contraction of areas in phase space,
   (c) Limit cycles, Hopf bifurcations, excitability
   (d) Power spectra, autocorrelations, Poincaré sections, maps (e.g., $x_{k+1} = f(x_k)$). Phase space reconstruction.

(e) Fluid dynamics and Rayleigh-Bénard (thermal) convection.

2. Deterministic chaos in low-dimensional systems.

   (a) Strange attractors. Sensitivity to initial conditions. Lorenz attractor, Hénon attractor, etc.

   (b) Quantifying chaos ("measuring the strangeness of strange attractors").

      i. Fractal dimension (how many "degrees of freedom"?; dynamics becomes geometry).

      ii. Lyaponov exponents (How sensitive to initial conditions?).

   (c) Transitions to chaos, scaling and universality.

      i. Period doubling. Oscillations of successively longer periods 2^n occur when the control parameter has value μ_n, with

      $$(\mu_\infty - \mu_n) \propto \delta^{-n}$$

      where the system is chaotoc at μ_∞ and $\delta = 4.669...$ is *universal*.

      ii. Intermittency.

      iii. Quasiperiodicity.

   The second half of the course stresses the relations between pde's, ode's, and discrete mapping. We shall see that much of the complexity of non-linear pde's is contained in simple 1-D maps!

3. Remaining time (if any): physical models of scale invariance (fractals) in nature.

1.6 Course material

Strongly recommended: Strogatz [2]. Beautifully written. Only a few lectures will follow it in detail, but nearly all the subjects we cover are addressed in the book. PDF is downloadable from the MIT Library.

All of our lectures will be accompanied by detailed lecture notes that will be posted to the Canvas website before or shortly after the lecture.

1.7 Students

Who should take this course?

- Scientists and engineers who desire to learn how and why nonlinearity manifests itself in natural systems.
- Mathematicians who seek a scientific, physical, and phenomenological inspiration for the further development of mathematical theory.
- Anyone interested in how one "does science" with computers.

1.8 Handouts and further reading

You should have

- Guidelines
- Syllabus
- Problem Set 1. Much easier than usual, just to get started, and an opportunity to resolve now any technical problems with Python or Matlab.

Also: For general background, *read Chapter 1 of Strogatz*.

References

1. Campbell, D. K. Nonlinear science. *Los Alamos Science* **15,** 218–262 (1987).
2. Strogatz, S. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (CRC Press, 2018).

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