

## CHAPTER 2

### FLOW PAST A SPHERE I: DIMENSIONAL ANALYSIS, REYNOLDS NUMBERS, AND FROUDE NUMBERS

#### INTRODUCTION

**1** Steady flow past a solid sphere is important in many situations, both in the natural environment and in the world of technology, and it serves as a good reference case for extension to more complicated situations, involving unsteady flows and/or nonuniform flows and/or nonspherical bodies. It is also an excellent starting point for development of a number of important principles and techniques that are essential for later development in these notes. In particular, I hope to be able to convince you of the importance and utility of careful dimensional reasoning about flows of fluids.

**2** You can think in terms of fluid flowing past a stationary sphere, or of a sphere moving through stationary fluid. The two cases are almost, but not quite, equivalent. And in the latter case you could imagine the sphere being moved through the fluid in three different ways: fastened to a rigid strut, or towed with a flexible line, or pulled downward through the fluid under its own weight. For now, do not worry about these distinctions; just view the fluid from the standpoint of the sphere. I will return to the differences briefly later. For the sake of definiteness, assume here that the sphere is towed or pushed through still fluid. All that is said here about the flow is then with reference to a point fixed relative to the moving sphere.

**3** Just from considerations of space and motion, it is clear that the approaching fluid must both move faster and be displaced laterally as it flows past the sphere. On the other hand, the no-slip condition requires that the fluid velocity be zero everywhere at the surface of the sphere; this implies the existence of gradients (that is, spatial rates of change) of velocity, very sharp under some conditions, at and near the surface of the sphere. These velocity gradients produce a shear stress on the surface of the sphere; see Equation 1.8. When summed over the surface, the shear stress exerted by the fluid on the sphere represents the part of the total drag force on the sphere called the *viscous drag*. Your intuition probably tells you (correctly in this case) that the pressure of the fluid, the normal force per unit area, is greater on the front of the sphere than on the back. The sum of the pressure forces over the entire surface of the sphere represents the other part of the drag force, called the *pressure drag* or *form drag*. You will see later that the relative importance of viscous drag and pressure drag, as well as the qualitative flow patterns and the distance out into the fluid the sphere makes its presence felt, are greatly different in different ranges of flow.

**4** You can see now that even in such a seemingly simple flow as the passage of a steady and uniform approach flow around a smooth sphere there is a great variation in flow phenomena. Complexity of this kind in deceptively simple

flows is common in fluid dynamics; you need to be on your guard against theorizing about phenomena of fluid flow without the ground truth of experiment and observation.

### WHICH VARIABLES ARE IMPORTANT?

**5** Think first about the resultant drag force  $F_D$  exerted on the sphere by the fluid (Figure 2-1). To account fully for the value assumed by  $F_D$  for a given sphere in a given fluid, we have to specify the values of certain other variables. (This carefully phrased sentence should not be interpreted as implying that  $F_D$  is necessarily the “dependent variable” in the problem; for a sphere settling under its own weight, it is more natural to think of  $F_D$  as an independent variable and settling velocity as the dependent variable. What is important here is that there

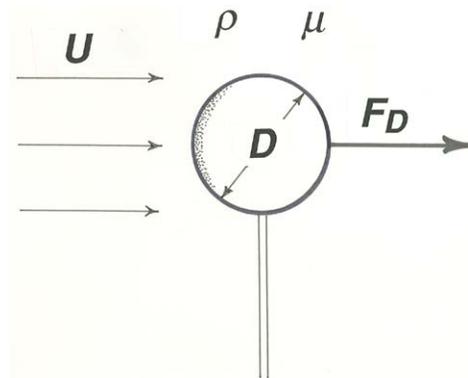


Figure 2-1. The drag force  $F_D$  on a sphere moving relative to a viscous fluid.

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is a one-to-one correspondence between the values of  $F_D$  and the values of those other variables, irrespective of their dependence or independence. That said, however, for convenience I will refer to such variables as independent variables.) The velocity  $U$  of the sphere relative to the fluid is important because it affects the shear in the fluid near the surface of the sphere, and therefore by Equation 1.9 the shear stress. Sphere diameter  $D$  is important for the same reason. Viscosity  $\mu$  is important because it determines the shear force associated with a given rate of shear. Fluid density  $\rho$  must also be included, because the forces associated with the accelerations in the fluid depend upon  $\rho$ : the response of a body to a force exerted on it depends on the mass of the body; that is the essence of Newton’s second law. If the sphere is in steady motion far from solid walls or a free surface, you can assume that no other variables are important. So

$$F_D = f(U, D, \rho, \mu) \quad (2.1)$$

where  $f$  is some function with one or more terms involving the four independent variables (Figure 2-1). (I will often use the same symbol  $f$  for unrelated functions. In Chapter 4,  $f$  is also used for a quantity called the friction factor.)

**6** You might reasonably ask why neither sphere density nor acceleration of gravity are on the list. These are relevant only if the sphere settles under its own weight, and then only because they determine the weight of the sphere, to which  $F_D$  is then equal after a steady state of settling is attained. Variables that enter the problem only by their effect on other variables already on the list and not because of some separate effect need not be included in the analysis. And there is no reason to think that either of these has any such significance.

**7** If we are lucky in problems like this, we can use theory to derive an analytical form for the function in Equation 2.1 that agrees well with observation. If not, we have to attempt a numerical solution or rely solely on experiment. For flow past a sphere there is indeed an analytical solution, described later in this chapter, that agrees beautifully with experimental data, but it holds over only a limited range of the independent variables; over the rest of the range we can obtain the function by experiment, as is commonly the case in problems of flow of real fluids. With flow past the sphere as an example we need to consider how we can best go about organizing both data and thought by resorting to dimensional reasoning.

### SOME DIMENSIONAL REASONING, AND ITS CONSEQUENCES

**8** Like every physically correct equation, Equation 2.1 must represent equality not only of *magnitudes* but also of *dimensions*. In most mechanical systems three basic dimensions are needed to express forces, motions, and system properties; these are usually taken to be mass (M), length (L), and time (T). So whatever the form of the term or terms on the right side of Equation 2.1, the variables  $U$ ,  $D$ ,  $\rho$ , and  $\mu$  must combine in such a way that each term has the dimensions of force, because the left side has the dimensions of force. The following list gives the dimensions of each of the five variables involved in flow past a sphere, in terms of mass M, length L, and time T:

$F_D$	—	ML/T <sup>2</sup>
$U$	—	L/T
$D$	—	L
$\rho$	—	M/L <sup>3</sup>
$\mu$	—	M/LT

The only variable here whose dimensions are not straightforward is  $\mu$ ; the dimensions M/LT are obtained by use of Equation 1.8, by which  $\mu$  is defined.

**9** It is advantageous to rewrite equations like Equation 2.1 in dimensionless form. To do this, first make the left side dimensionless by dividing  $F_D$  by some

product of independent variables that itself has the dimensions of force. Using the list of dimensions above, you can verify that  $\rho U^2 D^2$  has the dimensions of force:

$$\rho U^2 D^2 \text{ ——— } (M/L^3)(L/T)^2(L)^2 = ML/T^2$$

So dividing the left side of Equation 2.1 by  $\rho U^2 D^2$  makes the left side of the equation dimensionless. The result,  $F_D/\rho U^2 D^2$ , can be viewed as a dimensionless form of  $F_D$ . That leaves the right side of Equation 2.1 to be made dimensionless. There is one and only one way the four variables  $U$ ,  $D$ ,  $\rho$ , and  $\mu$  can be combined into a dimensionless variable, namely  $\rho U D/\mu$ :

$$\rho U D/\mu \text{ (M/L}^3\text{)(L/T)(L)/(M/LT) } \quad M, L, T \text{ cancel}$$

(That statement is not strictly true—but all the other possibilities are just  $\rho U D/\mu$  raised to some power, and they are not independent of  $\rho U D/\mu$ .) So whatever the form of the function  $f$ , the right side of the dimensionless form of Equation 2.1 can be written using just one dimensionless variable:

$$\frac{F_D}{\rho U^2 D^2} = f\left(\frac{\rho U D}{\mu}\right) \quad (2.2)$$

**10** Equation 2.2 is an equivalent but dimensionless form of Equation 2.1. The great advantage of the dimensionless equation is that it involves only two variables—a dependent dimensionless variable  $F_D/\rho U^2 D^2$  and an independent dimensionless variable  $\rho U D/\mu$ —instead of the original five. Think of the enormous saving in effort this implies for an experimental program to characterize the drag force. If you had to measure  $F_D$  as a function of each one of the four variables while holding the other three constant, you would generate mountains of data and graphs. But Equation 2.2 tells you that  $U$ ,  $D$ ,  $\rho$ , and  $\mu$  need only be varied so as to make  $\rho U D/\mu$  vary. All of the experimental points for  $F_D/\rho U^2 D^2$  obtained by varying  $\rho U D/\mu$  should plot as a curve in a two-dimensional graph with these two variables along the axes. Whatever the values of  $U$ ,  $D$ ,  $\rho$ , and  $\mu$ , all possible realizations of flow past a sphere are expressed by just one curve. This curve is shown in Figure 2-2 together with some of the experimental points that have been used to define it. The physics behind the curve is discussed in Chapter 3, after more background in the principles of fluid dynamics. And you could find the curve by varying only *one* of the four variables  $U$ ,  $D$ ,  $\rho$ , and  $\mu$ —although you may not be able to get a very wide range of values of  $\rho U D/\mu$  by varying only one of those variables. A fairly small number of experiments involving values of the original independent variables that combined to span a wide range of  $\rho U D/\mu$  would suffice to characterize all other possible combinations of independent variables. This is because each point in the dimensionless graph represents a great many different possible combinations of the original variables—an infinity of these, in fact. You thus gain a far-reaching predictive capability on the basis of relatively little observational effort.

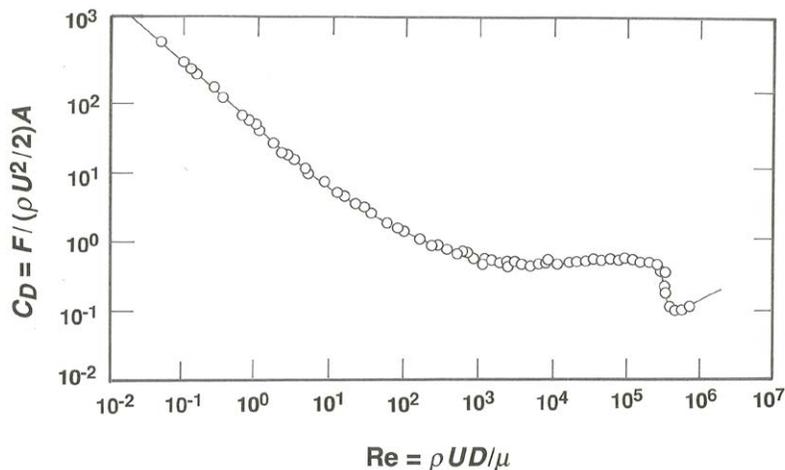


Figure 2-2. Plot of dimensionless drag force vs. Reynolds number for flow of a viscous fluid past a sphere. The dimensionless drag force is expressed in the form of a conventionally defined drag coefficient rather than as the dimensionless drag force  $F_D$ ; see further in the text. Experimental points are from several sources, and are somewhat generalized. Some of the data points are from settling of a sphere through a still fluid, and others are from flow past a sphere held at rest. For a more detailed plot, see, for example, Schiller (1932).

**11** A skeptic might find all this to be too good to be true. But the fact is that this is how things work, and the analysis of flow past a sphere is just one good example. A note of caution is in order, however. It is prudent to vary as many of the variables over as wide a range as possible; this does not take an enormous number of observations, and it is a check on the correctness of your analysis. You will see below in more detail that if there is a larger number of important variables than you think, your data points would form a scattered band rather than a single curve. Then if you varied just one variable to try to find the curve, you would indeed get a curve, but it would not be the curve you were after; you would be missing the scatter that would manifest itself if you varied the other variables as well.

**12** Several notes are in order here:

(1) Variables of the form  $\rho U D / \mu$  are called **Reynolds numbers**, usually denoted by  $Re$ . Whenever both density and viscosity are important in a problem and both a length variable and a velocity are involved, a Reynolds number can be formed and used. There are thus many different Reynolds numbers, with different length and velocity variables depending on the particular problem. You will encounter others in later chapters.

(2) For the steady flow we have assumed, the variables  $U$ ,  $D$ ,  $\rho$ , and  $\mu$  characterize not only everything about the distributions of shear stress and

pressure over the entire surface of the sphere, which add up to  $F_D$ , but also the distributions of shear stress, pressure, and fluid velocity at every point in the surrounding fluid. Because  $\rho U D / \mu$  replaces these four variables on the right side of Equation 2.2, the same can be said of the Reynolds number. Anything about forces and motions you might want to consider can be viewed as being specified completely by the Reynolds number.

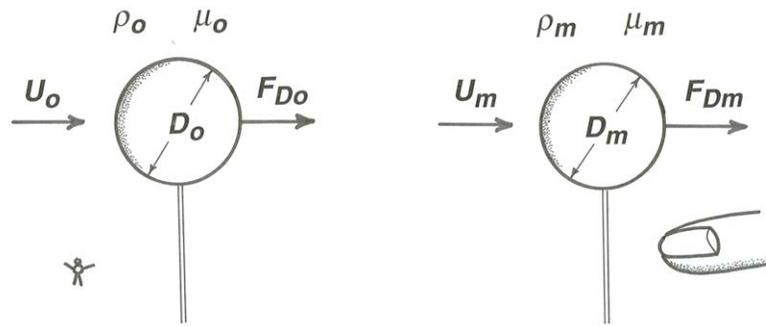


Figure 2-3. An example of scale modeling: using flows around a small sphere to model flow around a large sphere. (The object in the lower right is supposed to be someone's fingertip.)

(3) There is a further important consequence of the fact that each point on the curve of  $F_D / \rho U^2 D^2$  vs.  $\rho U D / \mu$  represents an infinity of combinations of  $U$ ,  $D$ ,  $\rho$ , and  $\mu$ . Suppose that you wanted to find the drag force exerted by a certain flow on a sphere that is too large to fit into your laboratory or your basement. You could work with a much smaller sphere by adjusting the values of  $U$ ,  $\rho$ , and  $\mu$  so that  $\rho U D / \mu$  is the same as in the flow in question past the large sphere (Figure 2-3). Then from the curve in Figure 2-2 the value of  $F_D / \rho U^2 D^2$  is also the same, and from it you could find the drag force  $F_D$  on the large sphere by substituting the corresponding values of  $U$ ,  $D$ , and  $\rho$ . Or, on the other hand, you could study the flow around a very small sphere by use of a much larger sphere, with the same complete confidence in the results (Figure 2-3). This is the essence of **scale modeling**: the study of one physical system by use of another at a smaller or larger physical scale but with variables adjusted so that all forces and motions in the two systems are in the same proportions. Figure 2-3 shows how you might use flow around a small sphere with diameter  $D_m$  to model flow around a much larger sphere with diameter  $D_o$ . You would have to adjust the flow velocities  $U_m$  and  $U_o$ , as well as the fluid viscosities  $\mu_m$  and  $\mu_o$  and the fluid densities  $\rho_m$  and  $\rho_o$ , so that the Reynolds number  $Re_m$ , equal to  $\rho_m U_m D_m / \mu_m$ , in the model is the same as the Reynolds number  $Re_o$ , equal to  $\rho_o U_o D_o / \mu_o$ , in the large-scale flow. Then all forces and motions are in the same proportion in the two flows, and, specifically, the dimensionless drag force, or the drag coefficient, is the same in the two flows. Despite the great difference in physical scale, both of the flows are represented by

the same point on the graph of drag coefficient vs. Reynolds number, so anything about the two flows, provided only that it is expressed in dimensionless form, is the same in the two flows. Each point on the curve of  $F_D/\rho U^2 D^2$  vs.  $\rho U D/\mu$  represents an infinite number of possible experiments, each of which is a scale model of all the others!

(4) In Figure 2-2 the dimensionless drag force is written in a conventional form that is slightly different from that derived above:  $F_D/(\rho U^2/2)A$ , where  $A$  is the cross-sectional area of the sphere, equal to  $\pi D^2/4$ . This differs from  $F_D/\rho U^2 D^2$  by the factor  $\pi/8$ , but its dimensions are exactly the same. It is usually called a **drag coefficient**, denoted by  $C_D$ ; you can see why that term came about by writing

$$F_D = C_D \frac{\rho U^2}{2} A \quad (2.3)$$

where the factor  $(\rho U^2/2)A$  on the right side has dimensions of force. The functional relationship between dimensionless drag force and Reynolds number in Equation 2.2 can be written in an entirely equivalent form using  $C_D$ :

$$C_D = \frac{F_D}{\frac{\rho U^2}{2} A} = f\left(\frac{\rho U D}{\mu}\right) \quad (2.4)$$

(5) There are alternative versions of the dependent dimensionless variable. Dividing by  $\rho U^2 D^2$  is not the only way to nondimensionalize  $F_D$ . You can check for yourself that  $F_D/\mu U D$ ,  $\rho F_D/\mu^2$ , and  $F_D/\mu U$  are other possibilities, obtained by combining  $F_D$  with the four variables  $\rho$ ,  $\mu$ ,  $U$ , and  $D$  taken three at a time. (You will see in the next section how to derive such variables.) Sometimes, as in the last two cases, one of the variables drops out; this happens when M or L or T appears in only one of the four variables chosen. Any of these three alternative dependent dimensionless variables would serve just as well as  $F_D/\rho U^2 D^2$  to represent the data. You will see below, however, that sometimes one is more revealing than the others.

## HOW TO CONSTRUCT DIMENSIONLESS VARIABLES

**13** You may be wondering about how you could have constructed the dimensionless variable  $\rho U D/\mu$  on your own instead of having it presented to you. Start with a very general product  $\rho^a U^b D^c \mu^d$ . The exponents  $a$  through  $d$  have to be adjusted so that the M, L, and T dimensions of the product cancel out. One of the exponents can be chosen arbitrarily, say  $d = 1$ , but then  $a$ ,  $b$ , and  $c$  have to be adjusted by solving three equations, one each for M, L, and T, expressing the condition that the product be dimensionless. Using length as an example, you can see from the list of dimensions above that length enters into  $\rho$  to the power -3, into  $U$  to the power +1, into  $D$  to the power +1, and into  $\mu$  to the power -1. So for

the length dimension to cancel out of  $\rho^a U^b D^c \mu$ , the following condition must be met:  $-3a + b + c - 1 = 0$ . (Keep in mind that we have already chosen  $d$  to be 1.) Two more conditions, one for M and one for T, give three linear equations in the three unknowns  $a$ ,  $b$ , and  $c$ :

$$\begin{array}{rcccc} -3a & +b & +c & -1 & = 0 & \text{(for L)} \\ +a & & & +1 & = 0 & \text{(for M)} \\ & -b & & -1 & = 0 & \text{(for T)} \end{array} \quad (2.5)$$

The solution is  $a = -1$ ,  $b = -1$ ,  $c = -1$ , so the product takes the form  $\mu/\rho UD$ . This is the inverse of the Reynolds number introduced above. If  $d$  had been taken as -1 at the outset, the result would have been the Reynolds number itself.

### WHAT IF YOU CHOOSE THE WRONG VARIABLES?

**14** What would be the consequences of including an irrelevant variable in analyzing the dimensional structure of a problem like that of flow past a sphere? Suppose, contrary to fact but just for the sake of discussion, that viscosity is not important in determining  $F_D$ . Then the functional relationship for  $F_D$  would be

$$F_D = f(U, D, \rho) \quad (2.6)$$

As before, you can start to make this equation dimensionless by forming the same dimensionless drag force  $F_D/\rho U^2 D^2$  on the left-hand side. But how about the right-hand side? The three variables  $U$ ,  $D$ , and  $\rho$  cannot be combined to form a dimensionless variable, because there is not enough freedom to adjust exponents to make a product  $U^a D^b \rho^c$  dimensionless; this should be clear from the formal procedure described above for obtaining  $\rho UD/\mu$ . Then what takes the place of the Reynolds number on the right side? The answer is that the right side must be a numerical *constant*: there *is* no independent dimensionless variable. So if  $\mu$  were not important in flow past a sphere, the dimensionless force

$F_D/\rho U^2 D^2$  would be a constant rather than a function of the Reynolds number. To generalize: if one original variable is eliminated from the problem, one dimensionless variable must be eliminated as well. In a graph of  $C_D$  vs.  $Re$  the experimental points would fall along a straight line parallel to the  $Re$  axis, as shown schematically in Figure 2-4. Now look back at the actual graph of  $C_D$  vs.  $Re$  in Figure 2-2. Over a wide range of Reynolds numbers from about  $10^2$  to greater than  $10^5$ ,  $C_D$  is nearly independent of the Reynolds number. Because  $\mu$  is the only variable that appears in the Reynolds number but not in  $C_D$ , this tells you that  $\mu$  is indeed not important in determining  $F_D$  at large  $Re$ . The reasons for this are discussed in Chapter 3.

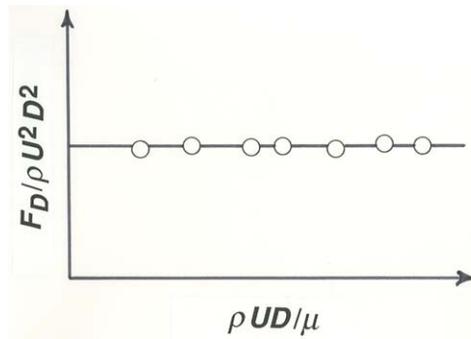


Figure 2-4. What the plot of dimensionless drag force vs. Reynolds number for flow around a sphere would look like if the viscosity were not important.

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**15** Now you can see why there is some practical advantage to using  $F_D/\rho U^2 D^2$  as the dependent dimensionless variable. The other three mentioned above contain  $\mu$ , and so in a plot of any one of them against  $\rho U D/\mu$  the segment of the curve for which  $\mu$  is not important would plot as a sloping line rather than as a horizontal line, and the unimportance of  $\mu$  would not be as easy to recognize.

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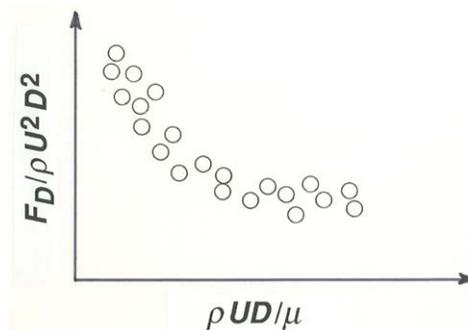


Figure 2-5. What the plot of dimensionless drag force vs. Reynolds number for flow around a sphere towed near a solid wall in a still body of water would look like if the distance of the sphere from the wall is not held constant from trial to trial.

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**16** You should also consider the consequences of omitting an important variable from consideration. For example, if you had not been careful to keep the sphere well away from the wall of the vessel containing the fluid, you would find (Figure 2-5) that the experimental points plot in a scattered band around the curve of  $C_D$  vs.  $Re$  in Figure 2-2. This tells you that some other variable is important in determining  $F_D$  and that you have inadvertently let it vary—assuming, of course, that your measurements are free of errors in the first place. The obvious culprit is

$y$ , the distance of the center of the sphere from the wall (Figure 2-6), because the proximity of the sphere to the solid wall distorts the pattern of flow around the sphere and thus changes the fluid forces on the sphere to some extent. With  $y$  included in the analysis, the functional relationship for  $F_D$  is of the form

$$F_D = f(U, D, \rho, \mu, y) \quad (2.7)$$

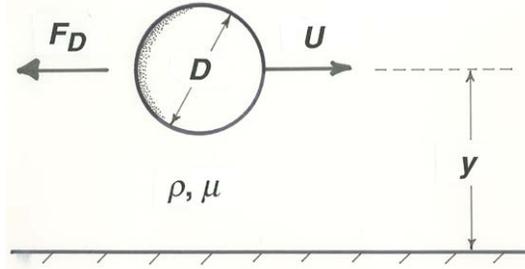


Figure 2-6. Towing a sphere parallel to a nearby solid planar wall.

**17** In nondimensionalizing Equation 2.7 you should again expect to have a dimensionless drag force on the left and the Reynolds number on the right. But what happens to the new variable  $y$ ? You can use it to form one more independent dimensionless variable, in the same way you formed the Reynolds number. There has to be at least one other such variable, because  $y$  has to appear somewhere on the right side of the nondimensionalized version of Equation 2.7. A natural choice for this new variable is  $y/D$  (or  $D/y$ ). You could instead form another Reynolds number,  $\rho U y / \mu$ . But only two of the three variables  $\rho U D / \mu$ ,  $\rho U y / \mu$ , and  $y/D$  are independent of each other: addition of one new independent variable to the problem adds only one new independent dimensionless variable. It is also worth pointing out that you can arrive at the second Reynolds number,  $\rho U y / \mu$ , by multiplying the first,  $\rho U D / \mu$ , by the new dimensionless variable  $y/D$ . This is an illustration of the principle that you can always replace a dimensionless variable in a set of dimensionless variables by another one formed by multiplying or dividing it by one of the others, or with some power or root of one of the others. So in dimensionless form Equation 2.7 is then

$$\frac{F_D}{\rho U^2 D^2} = f\left(\frac{\rho U D}{\mu}, \frac{y}{D}\right) \quad (2.8)$$

**18** The function in Equation 2.8 would plot as a curved surface in a three-dimensional graph with  $C_D$ ,  $Re$ , and  $y/D$  along the axes (Figure 2-7). The two

planes perpendicular to the  $y/D$  axis in Figure 2-7 show the range over which  $y/D$  varied in your experiments without your realizing that it is important. The projection of the segment of the surface between these two planes onto the  $C_D$ - $Re$  plane is the band in which your experimental points would fall. The intersection of the surface with the plane  $y/D = 0$ , also shown on the projection, represents the curve you would have gotten if you had always kept the sphere very far away from the wall; it is the same as the curve in Figure 2-2.

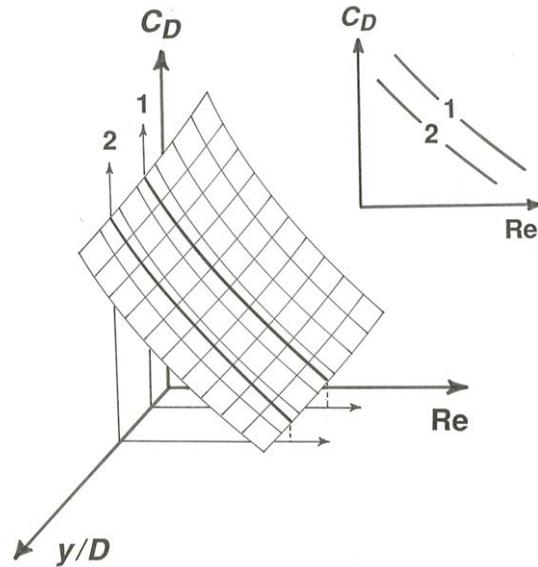


Figure 2-7. For towing of a sphere parallel to a nearby solid planar wall, data from a large number of trials would plot as a surface in a three-dimensional graph of drag coefficient, Reynolds number, and ratio of distance from wall to sphere diameter. The graph in the upper right shows, in a plot of drag coefficient vs. Reynolds number, two curves corresponding to two different values of the ratio of distance from wall to sphere diameter. These curves are the intersections of the full surface with planes parallel to the  $C_D$ - $Re$  plane.

**19** You could carry the analysis one step further by moving the sphere horizontally just beneath the free surface of a liquid at rest in a gravitational field (Figure 2-8). Of importance now is not only the distance  $y$  of the sphere below the free surface but also the acceleration of gravity  $g$ : if the movement of the sphere distorts the free surface, unbalanced gravity forces would tend to flatten the surface again, and surface gravity waves may be generated. Then

$$F_D = f(U, D, \rho, \mu, y, g) \quad (2.9)$$

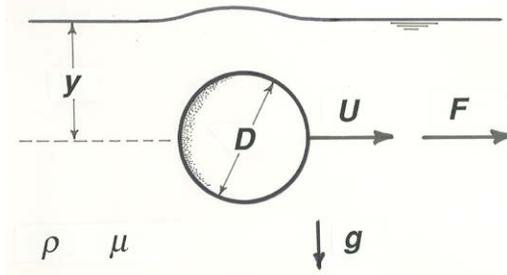


Figure 2-8. Towing a sphere horizontally through a still liquid, not far below the free surface of the liquid.

This adds still another independent dimensionless variable, and that variable must include  $g$ . There are five possibilities:  $\mu g / \rho U^3$ ,  $\rho^2 g D^3 / \mu^2$ ,  $\rho^2 g y^3 / \mu^2$ ,  $U^2 / g D$ , and  $U^2 / g y$ , plus obvious variants obtained by inversion and exponentiation. (You could try constructing these by combining  $U$ ,  $\rho$ ,  $\mu$ ,  $D$ , and  $y$  three at a time with  $g$  and going through the procedure described above for  $Re$ . You would also get  $y/D$  again in the process.) Any one of these five would suffice to express the effect of  $g$  on the drag force. Again only one is independent, because the others can all be obtained by combining that one (whichever you choose) with either  $\rho U D / \mu$  or  $y/D$ . It would be conventional, in a problem like this, to use  $U/(gy)^{1/2}$  as the added independent variable. The dimensionless form of Equation 2.9 is then

$$\frac{F_D}{\rho U^2 D^2} = f\left(\frac{\rho U D}{\mu}, \frac{U^2}{g y}, \frac{y}{D}\right) \quad (2.10)$$

The square root of a variable like  $U^2/gy$  or  $U^2/gD$ , with a velocity, a length variable, and  $g$ , is called a **Froude number**, usually denoted by  $Fr$ . It is natural, although not essential, to use  $U^2/gy$  here because then each of the four dimensionless variables in the functional relationship can be viewed as being formed by combining  $F_D$ ,  $\mu$ ,  $y$ , and  $g$  in turn with the three variables  $\rho$ ,  $U$ , and  $D$ ; see the following paragraph for details.

**20** The function in Equation 2.10 would plot as a four-dimensional “surface” in a graph of  $C_D$  vs.  $Re$ ,  $Fr$ , and  $y/D$ . It is difficult to visualize such a graph. A good substitute would be to plot a three-dimensional graph for each of a series of values of one of the independent dimensionless variables. The trouble is that there is an infinite number of these three-dimensional graphs. (I remember once reading somewhere that to express graphically the relationship between two variables you need a page, and to express the relationship among three variables you need a book of pages, and to express the relationship among four variables you need a library of books. For five variables you would need a world of libraries!)

**21** Suppose that you had realized at the outset that all seven variables in Equation 2.9 are important in the problem. The systematic way of obtaining four dimensionless variables all at once is just an extension of the method described in an earlier section for obtaining the Reynolds number. Form four products by choosing three of the seven variables (the “repeating” variables) to be those raised to the exponents  $a$ ,  $b$ , and  $c$  and using each of the remaining four variables in turn as the one that is raised to the exponent 1 (or to any other fixed exponent, for that matter). You can verify for yourself that if you choose  $\rho$ ,  $U$ , and  $D$  as the three repeating variables, the four products  $\rho^a U^b D^c F_D$ ,  $\rho^a U^b D^c \mu$ ,  $\rho^a U^b D^c y$ , and  $\rho^a U^b D^c g$  would produce the four dimensionless variables in Equation 2.10, except that  $U^2/gD$  appears instead of  $U^2/gy$ . It turns out that for this procedure to work, the constraints on the choice of the three repeating variables are that (1) among them they include all three dimensions M, L, T, and (2) they be dimensionally independent of each other, in the sense that you cannot obtain the dimensions of any one by multiplying together the dimensions of the other two after raising them to some exponents. These constraints just ensure that you get solvable sets of simultaneous equations.

### “DIMENSIONAL ANALYSIS”

**22** Most kinds of fluid flow that are important in natural environments do not lend themselves to analytical solutions, even when no sediment is moved, so experiment and observation are a valuable way to learn something about them. I have expatiated upon dimensionless variables and their use in expressing experimental results because this sort of analysis, usually called dimensional analysis, is so useful in dealing with problems of fluid flow and sediment movement. *Dimensional analysis* is a way of getting some useful information about a problem when you cannot obtain an analytical solution and may not even know anything about the form of the solution, but you have some ideas about important physical effects or variables. You will encounter many examples of its use in later chapters.

**23** Suppose that you are dealing with a fluid-flow problem that can be simplified somehow, perhaps in geometry or in time variability, to be manageable but still representative. Use your experience and physical intuition to identify the important variables. Form a set of dimensionless variables by which the observational results can be expressed. This represents the most efficient means of dealing with experimental data, and it usually makes it possible to get some idea of the ranges in which certain physical effects are important or unimportant. Do not worry too much about guessing wrong about important variables; the example of flow past a sphere shows how you can find out and change course.

**24** The number of dimensionless variables equivalent to a given set of original variables is given by the *Pi theorem*, also called Buckingham’s theorem. By the Pi theorem, the number of dimensionless variables corresponding to a number  $n$  of original variables that describe some physical problem is equal to  $n - m$ , where  $m$  is the number of dimensions by which the problem must be expressed. If you want to go back to the original source of the proofs (the

theorem was not proved in the foregoing material, just demonstrated), see Buckingham (1914, 1915).

## SIGNIFICANCE OF REYNOLDS NUMBERS AND FROUDE NUMBERS

**25** Some further insight into the significance of Reynolds numbers and Froude numbers is afforded by showing that dimensionless variables of this form always arise in problems involving viscous forces and gravity forces. But first I want to make sure you know what an equation of motion is.

**26** The *equation of motion* for some body of matter, whether solid or fluid, whether discrete or continuous, is just Newton's second law written for that body. You write out the sum of all the forces acting on the body and set that sum equal to the mass times the acceleration. The equation of motion for a continuous medium like a fluid comes out to be a differential equation. Why? Because to derive the equation you have to write it for some element of fluid with finite volume, and then watch what happens to the equation as the volume element shrinks to a point.

**27** Think about the balance of forces on some small element of fluid in any fluid-flow problem (for example, that of a sphere moving near a free surface) that involves fluid shear forces and also gravity forces that are not simply balanced out by hydrostatic pressure. Whatever the exact nature of the problem, Newton's second law must hold for this small element of fluid, so we can write for it a general equation of motion in words:

$$\begin{aligned} &\text{viscous force} + \text{gravity force} + \text{any other forces} \\ &= \text{rate of change of momentum} \end{aligned} \quad (2.11)$$

All of the terms in this equation have the same dimensions, so we can divide all the terms by any one of them to obtain an equation with all terms dimensionless. Dividing by the term on the right,

$$\begin{aligned} &\frac{\text{viscous force}}{\text{ROC of momentum}} + \frac{\text{gravity force}}{\text{ROC of momentum}} \\ &+ \frac{\text{other forces}}{\text{ROC of momentum}} = 1 \end{aligned} \quad (2.12)$$

**28** What will be the form of the first two dimensionless terms on the left side of Equation 2.12, in terms of representative variables that might be involved in any given flow problem? Assuming that there is some characteristic length variable  $L$  in the problem like a sphere size or flow depth, and some characteristic velocity  $V$  like the approach velocity in flow past a sphere or the mean velocity or surface velocity in flow in a channel, then the rate of change of momentum, which has dimensions of momentum divided by a characteristic time  $T$ , can be written as proportional to  $\rho L^3 V/T$ . (Remember that the mass can be expressed as density times volume and the volume as the cube of a length.) And this can further be

written  $\rho L^2 V^2$ , because velocity has the dimensions  $L/T$ . The viscous force is the product of the viscous shear stress and the area over which it acts. Area is proportional to the square of the characteristic length, and by Equation 1.9 the shear stress is proportional to the viscosity and the velocity gradient, so the viscous force is proportional to  $\mu(V/L)L^2$ , or  $\mu VL$ . The first term in Equation 2.12 is then proportional to  $\mu VL/\rho L^2 V^2$ , or  $\mu/\rho LV$ . This is simply the inverse of a Reynolds number. The Reynolds number in any fluid problem is therefore inversely proportional to the ratio of a viscous force and a quantity with the dimensions of a force, the rate of change of momentum, which is usually viewed as an “inertial force”.

**29** How about the second term in Equation 2.12? The gravity force is the weight of the fluid element, which is proportional to  $\rho g L^3$ . The second term is then proportional to  $\rho g L^3/\rho L^2 V^2$ , or  $gL/V^2$ . This is the square of the inverse of a Froude number. The square of the Froude number is therefore proportional to the ratio of a gravity force and a rate of change of momentum or an “inertial force”.

**30** This probably strikes you as not a very rigorous exercise—and indeed it is not. It is intended only to give you a general feel for the significance of Reynolds numbers and Froude numbers. At the expense of lengthening this chapter considerably, the general differential equation of motion for flow of a viscous fluid could be derived and then made dimensionless by introducing the same characteristic length and characteristic velocity, and a reference pressure as well. You would see that the Reynolds number and the Froude number then emerge as coefficients of the dimensionless viscous-force term and dimensionless gravity-force term, respectively. This is done especially lucidly by Tritton (1988, Chapter 7). The value of such an exercise is that then the magnitudes of the Reynolds number and Froude number tell you whether the viscous-force term or the gravity-force term in the equation of motion can be neglected relative to the mass-times-acceleration term. This is a productive way of simplifying the equation of motion to gain some insight into the physics of the flow.

**31** When you are deciding which set of dimensionless variables to work with in problems like that of flow past a sphere, introduced above, it makes sense to use dimensionless variables that have their own physical significance, like Reynolds numbers and Froude numbers. In later chapters, other dimensionless variables are introduced that represent ratios of two forces in specific problems.

## CONCLUSION

**32** Before you are confronted any further with the physics of flow past spheres, you need to be introduced to quite a bit more material on fluid flow. The first part of the next chapter, Chapter 3, is devoted to this material, before more on the topic of flow past spheres.

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