Definition 1: A random vector $\mathbf{X}$ has a *nondegenerate (multivariate) normal* distribution if it has a joint PDF of the form

$$f_X(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n|V|}} \exp\big(-(\mathbf{x}-\mu)V^{-1}(\mathbf{x}-\mu)^T/2\big),$$

for some real vector μ and for some positive definite matrix V.

Definition 2: A random vector $\mathbf{X}$ has a *(multivariate) normal* distribution if it can be expressed in the form

$$\mathbf{X} = D\mathbf{W} + \mu,$$

for some matrix D and some real vector μ, where $\mathbf{W}$ is a random vector whose components are independent $\mathcal{N}(0,1)$ random variables.

Definition 3: A random vector $\mathbf{X}$ has a *(multivariate) normal* distribution if for every real vector $\mathbf{a}^T$, the random variable $\mathbf{a}^T\mathbf{X}$ is normal random.

Theorem: Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ is multivariate normal, in the sense of Definition 2.

(a) For every i, X_i is normal, with mean μ_i.

(b) We define the *covariance matrix* $\text{Cov}(\mathbf{X},\mathbf{X})$ of $\mathbf{X}$ to be a matrix whose ijth entry is $\text{Cov}(X_iX_j)$. Then, $\text{Cov}(\mathbf{X},\mathbf{X}) = \text{DD}^\text{T}$.

(c) If C is a $m \times n$ matrix and $\mathbf{d}$ is a vector in $\Re^m$, then $Y = C\mathbf{X} + d$ is multivariate normal in the sense of Definition 2, with mean $C\mu + \mathbf{d}$ and covariance CDD^TC.

(d) If $|D| \neq 0$, then $\mathbf{X}$ is nondegenerate multivariate normal in the sense of Definition 1, with $V = DD^T$.

(e) The joint CDF F_X of $\mathbf{X}$ is completely determined by the mean and covariance of $\mathbf{X}$.

(f) The components of $\mathbf{X}$ are uncorrelated (diagonal covariance matrix) if and only if they are independent.

(g) If

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} V_X & V_{XY} \\ V_{YX} & V_Y \end{bmatrix},\right)$$

and $|V_Y| \neq 0$, then:

(i) $\mathbf{E}[\mathbf{X} \mid \mathbf{Y}] = \mu_X + V_{XY}V_{YY}^{-1}(\mathbf{Y} - \mu_Y)$.

(ii) Let $\tilde{\mathbf{X}} = X - \mathbf{E}[X \mid Y]$. Then, $\tilde{\mathbf{X}}$ is independent of $\mathbf{Y}$.

(iii) $\text{Cov}(\tilde{\mathbf{X}}, \tilde{\mathbf{X}} \mid \mathbf{Y}) = \text{Cov}(\tilde{\mathbf{X}}, \tilde{\mathbf{X}}) = V_X - V_{XY}V_{YY}^{-1}V_{YX}$.