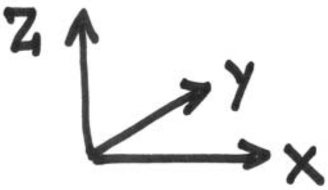
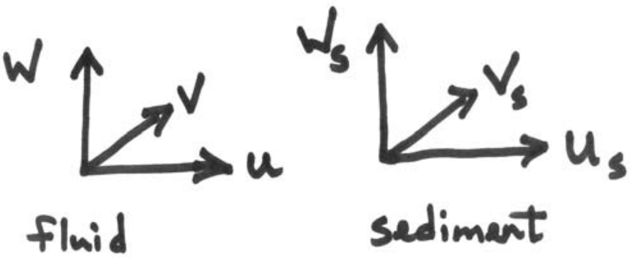
Coordinate System



All descriptions of fluid and sediment transport will be referenced to the following orthogonal coordinate system. In this system, ***x*** refers to the streamwise or down-slope direction, ***y*** refers to the cross-stream or cross-slope direction, and ***z*** refers to the vertical direction.



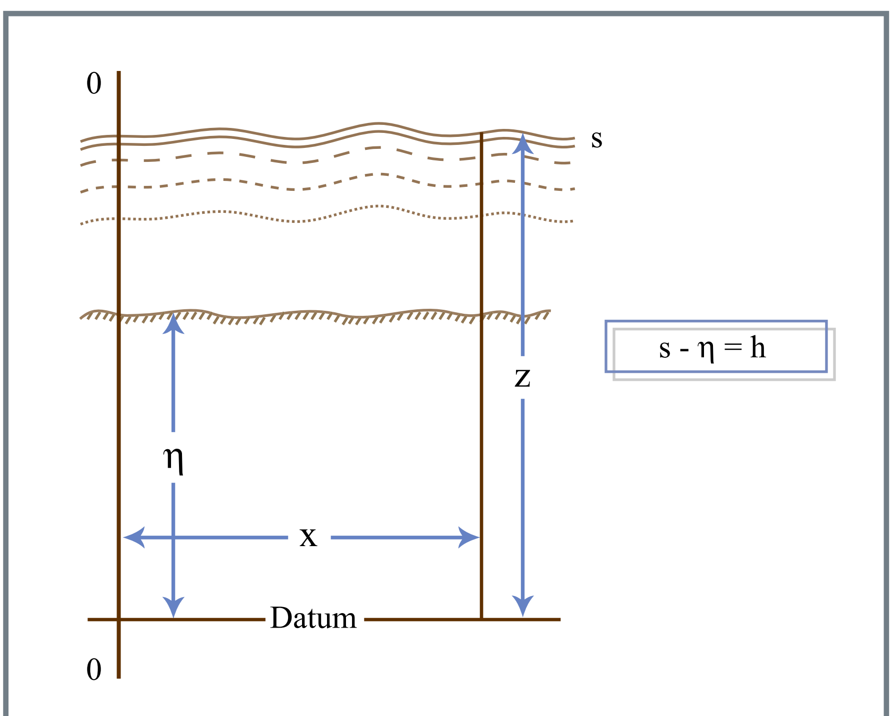
The fluid velocity component in the x direction will be referred to as ***u***, the fluid velocity component in the y direction will be referred to as ***v***, and the component in the z direction is ***w***. A subscript ***s*** will be added to each velocity component to distinguish the velocity of the sediment grains from the velocity of the fluid.

Vector Notation

$\underset{\sim}{X} = X_i = x + y + z$

$\underset{\sim}{U} = U_i = u + v + w$

Reference Frame

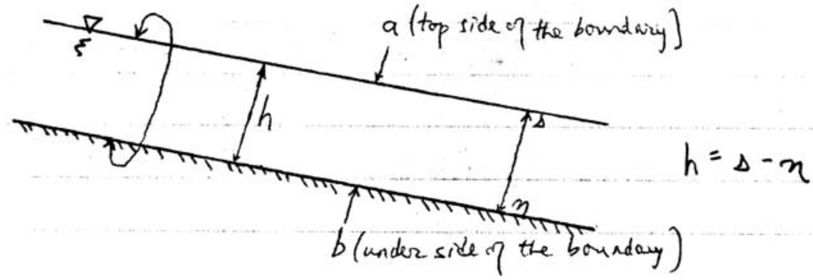


h = depth of flow

Figure by MIT OCW.

The Erosion Equation is derived by integrating the expression for mass conservation through the entire depth of the flow and applying the following boundary conditions…..

ε_s on the top side of the water surface = 0
u_s on the under side of the sediment bed = 0



$$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_{bed}}\left(\nabla \cdot \underset{\sim}{q}_s + \frac{\partial V_s}{\partial t}\right) = -\frac{1}{\varepsilon_{bed}}\left(\left(\frac{\partial q_{s_x}}{\partial x} + \frac{\partial q_{s_y}}{\partial y} + \frac{\partial q_{s_z}}{\partial z}\right) + \frac{\partial V_s}{\partial t}\right)$$

η = elevation of the uppermost layer of non-moving grains (units: m),
t = time (units: s),
ε_{bed} = concentration of sediment in the bed (1-porosity)
q_s = sediment flux or sediment discharge per unit width (units: m^2/s)
V_s = volume of sediment in motion per bed area (units: m)

The equation states that the rate of elevation change of the bed (i.e., erosion or deposition) is equal to the divergence or spatial change in the sediment flux plus the rate of change in suspended sediment (approximately V_s)

Useful Simplification of the Erosion Equation:
A. two-dimensional form.
B. V_s = small value, set to zero

$$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_{bed}}\left(\frac{\partial q_s}{\partial x}\right)$$

what does this mean?

$$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_{bed}}\left(\frac{\partial q_s}{\partial x}\right) = -\frac{1}{\varepsilon_{bed}}\left(\frac{\partial q_s}{\partial \tau_b}\right)\left(\frac{\partial \tau_b}{\partial x}\right)$$

$\left(\frac{\partial q_s}{\partial \tau_b}\right)$ is almost always positive, so the only way to go from erosion to deposition or vice versa is to change the sign of $\left(\frac{\partial \tau_b}{\partial x}\right)$

So **deposition** and **erosion** is primarily the consequence of a **spatial change** in **boundary shear stress**

An application of sediment conservation

Equalibrium shape: no shape change, migrating at constant speed C, 2-D case

by definition $$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + C\frac{\partial \eta}{\partial x} = 0$$

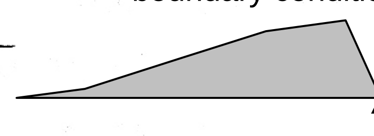
$$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_b}\frac{\partial q_s}{\partial x}$$ Erosion equation (neglecting $\frac{\partial V_s}{\partial t}$ term)

$$-\frac{1}{\varepsilon_b}\frac{\partial q_s}{\partial x} + C\frac{\partial \eta}{\partial x} = 0$$

integrate over x and rearrange terms

$$\eta = \underbrace{\left(\frac{1}{\varepsilon_b C}\right)}_{\text{constant}} q_s + \text{constant}$$

typically imposed boundary condition:



$\eta = 0$, $q_s = 0$, constant = 0

so if an equilibrium shape exists it has the same spatial structure as the sediment flux

$\langle \eta \rangle$ and $\langle q_s \rangle$ correlated

crest height equals $2\langle \eta \rangle \Rightarrow 2\langle q_s \rangle$

$$\langle q_s \rangle = \varepsilon_{bed} C \langle \eta \rangle = \varepsilon_{bed} C \frac{H}{2}$$

Simons, Richardson and Nordin (1965) Bedload equation for ripples and dunes

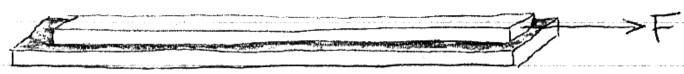
Fluid Mechanics as it Applies to Sediment Transport

STRESS IS A MEASURE OF FORCE ON THE INSIDE OF A CONTINUUM

STRAIN Is THE RESPONSE OF MATERIAL PARTICLES TO STRESS

CONSTITUTIVE EQUATIONS EXPRESS THE RELATIONSHIP BETWEEN STRESS & STRAIN (RATE)

Constitutive EQUATION FOR A NEWTONIAN FLUID



(stress on z face in x direction) $\tau_{zx} = \frac{F}{A_p}$

(strain rate) $\frac{\partial u}{\partial z} = \frac{u_p}{h_{(\text{depth of fluid})}}$ NO SLIP B.C. $\Rightarrow$ $\partial u = u_{plate} - u_{\text{lower boundary}} = u_p - 0 = u_p$

so, $\frac{\tau_{zx}}{\frac{\partial u}{\partial z}} = \frac{F/A_p}{\frac{u_p}{h}} = \mu$ (constant for a given material at a given temp)

dynamic viscosity

($\nu = \frac{\mu}{\rho}$, kinematic viscosity)

$$\boxed{\tau_{zx} = \mu \frac{\partial u}{\partial z}}$$

NEWTON'S VISCOUS LAW

μ has units of stress/strain rate $= Pa/(1/t) = N s/m^2 = Pa\,s$

$[m\,kg/s^2]\,s/m^2 = kg/m\,s$

water (20°C) $\eta = 10^{-3}\,Pa\,s$

asthenosphere $\eta \sim 10^{20}\,Pa\,s$

$\upsilon = \frac{\mu}{\rho} = \frac{[kg/m\,s]}{[kg/m^3]} = m^2/s$ (units of velocity) Kinematic viscosity

Conservation of Linear Momentum

so: $\rho \frac{d\underset{\sim}{u}}{dt} = \underset{\sim}{f_B} + \nabla \cdot \tilde{T}_{total} = \underset{\sim}{f_B} - \nabla p + \nabla \cdot \tilde{T}_{deviatoric}$

this component (∇p) expresses translation and compression

this component expresses rotation & pure shear

Navier - Stokes equation (early 1800's)

french engineer

english mathematician

$$\underbrace{\rho \frac{d\underset{\sim}{u}}{dt}}_{\text{acceleration of fluid}} = \underbrace{-\rho g}_{\text{body force}} \underbrace{- \nabla p}_{\text{pressure gradient}} + \underbrace{\mu \nabla^2 \underset{\sim}{u}}_{\text{diffusion of velocity}}$$

NAVIER-STOKES EQUATION (which is really nothing more than a modified expression of conservation of linear momentum)

Isotropic, incompressible, viscous Newtonian fluid, at constant temperature

and is commonly referred to as the ***Navier-Stokes Equation***, which is nothing more than a modified expression for the conservation of fluid momentum.

Scaling the Navier-Stokes Equation

For many geologically relevant flows all of the terms of the equation of motion are not needed to approximately describe the fluid motion.

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \vec{\nabla} \vec{V} = G + P + T$$

I time dependent

I space dependent

products of velocity components with their derivatives, making equation nonlinear (unless accelerations can be neglected)

simplifying equation

$$I_t / I_s + 1 = G / I_s + P / I_s + T / I_s$$

I = inertia = resistance of a body to a change in its acceleration

Navier-Stokes:
$$\rho\frac{\partial \underset{\sim}{u}}{\partial t} + \rho\underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = -\nabla p + \mu\nabla^2\underset{\sim}{u} - \rho\underset{\sim}{g}$$

$$\frac{\rho\frac{u_\infty}{T}}{\left(\rho\frac{u_\infty^2}{d}\right)} + \frac{\rho\frac{u_\infty^2}{d}}{\left(\rho\frac{u_\infty^2}{d}\right)} = -\frac{\frac{p}{d}}{\left(\rho\frac{u_\infty^2}{d}\right)} + \frac{\mu\frac{u_\infty}{d^2}}{\left(\rho\frac{u_\infty^2}{d}\right)} - \frac{\rho g}{\left(\rho\frac{u_\infty^2}{d}\right)}$$

$$\left[\ \frac{d}{\underset{\sim}{u}_\infty T} + 1 = -\frac{p}{\rho\left(\frac{\underset{\sim}{u}_\infty^2}{2}\right)} + \frac{\nu}{\underset{\sim}{u}_\infty d} - \frac{g\,d}{\underset{\sim}{u}_\infty^2}\ \right]$$

Strouhal #

Euler # (if $\underset{\sim}{u}_\infty^2$ divided by 2)

inverse of the Reynold's #

$(\text{Froude } \#)^{-\frac{1}{2}}$

Now divide by V^2/L

$$\left(\frac{L}{VT}\right)\frac{\partial\bar{v}'}{\partial t'} + \bar{v}'\cdot\bar{\nabla}\bar{v}' + \left(\frac{2\Omega L}{V}\right)\hat{k}\times\bar{v}' = \left(-\frac{\pi}{\rho V^2}\right)\bar{\nabla}'p' - \left(\frac{gL}{V^2}\right) + \left(\frac{\nu}{VL}\right)\bar{\nabla}'^2\bar{v}'$$

DIMENSIONLESS #s [CHARACTERISTIC LENGTHS OR MAGNITUDES]

$\frac{L}{VT} = S$: Strouhal # $= \frac{\text{acceleration}}{\text{inertia}}$

$\frac{VL}{\nu} = Re$: Reynolds # $= \frac{\text{inertia forces}}{\text{friction (viscous forces)}}$

$\frac{V^2}{gL} = F$: Froude # $= \frac{\text{kinetic energy}}{\text{gravitational potential energy}}$ $\frac{\text{inertia forces}}{\text{gravity forces}}$

$\frac{V}{2\Omega L} = R_o$: Rossby # $= \frac{\text{inertia}}{\text{coriolis}}$

$\frac{\Delta p \text{ (local pressure − freestream pressure)}}{\frac{1}{2}\rho V^2} = E_u$: Euler #

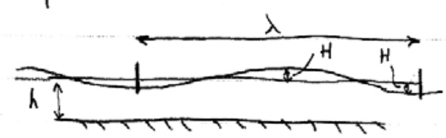
(Used to predict cavitation.)

Froude number: applicable to laminar and turbulent flows having a free surface OR Interface such that gravity forces play an important role in causing the flow.
(gh)^.5 = phase speed for shallow water surface wave (wavelength > water depth)
Fr < 1 waves can propagate both upstream and downstream, Tranquil flow.
Fr > 1 wave cannot propagate upstream, Shooting flow
Fr = 1 hydraulic jump, all upstream propagating waves are ‘stuck’ here

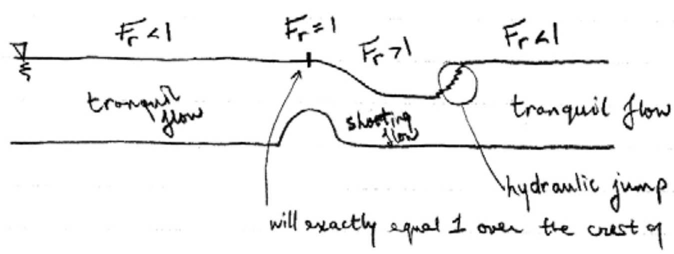
Froude number (Fr) is important in stream flow problems: $\frac{\langle u \rangle}{\sqrt{gh}}$

$\sqrt{gh} = C =$ phase speed of a long gravity wave

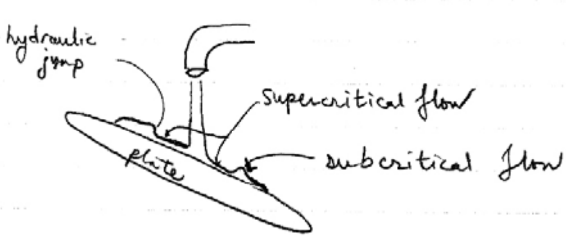
What is a long gravity wave?
$\lambda \gg h$
$H/h \ll 1$



If $F \ll 1$ waves can propogate upstream & downstream
$F \gg 1$ waves can propogate only downstream



$F_r = 1$ is the transition from tranquil to shooting flow



Reynolds Number: by looking at the size of the number you can see if inertia or friction is more important in the system.

Re <1 Laminar flow: stable to small disturbances (reversible deformation)

Re >> 1 Turbulent flow: unstable to small disturbances, stretching and twisting

In nature you always have disturbances, question is when do they decay versus grow?

Re < 500 laminar

Re > 500 turbulent (import natural flow, could through away laminar conditions, if not for boundary layers, length scale gets very small)

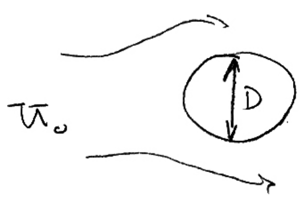
if $Re > 1000$, one can neglect the viscous forces & drop the $\mu \nabla^2 \underline{u}$ term out of N-S. equation

if $Re \sim 10^{-3}$, one can neglect convective accelerations & drop the $\rho \underline{u} \cdot \nabla \underline{u}$ term out of the N-S equation

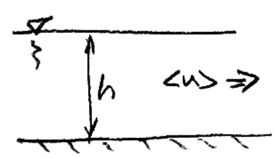
if $Re \sim 1$, one needs to keep both terms in the N-S equation

* This "scaling" allows one to get at the "essence" of the N-S. equation for a particular problem without having to solve for the entire equation.

Examples of scale velocity & length



$$R_D = \frac{U_o D}{\nu}$$



$$R = \frac{\langle u \rangle h}{\nu}$$

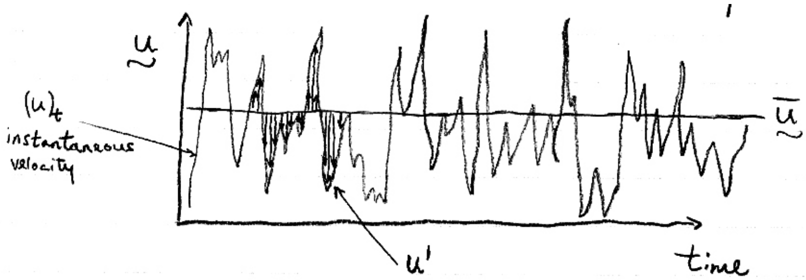
Remember, for channel flow problems: $u \approx \langle u \rangle$, characteristic length = h = depth (for uniform channel flow the only characteristic length is the depth)

$$Re = \frac{\langle u \rangle h}{\nu}, \quad F_r = \frac{\langle u \rangle}{\sqrt{gh}}$$

It is important to remember that Re and Fr are defined independently of one another.

TURBULENT FLOWS

Nonaveraged N-S equation describes the instantaneous velocity field. If a turbulent flow field is composed of a deterministic plus completely random part it is not possible to solve for this instantaneous field. Therefore the N-S equation must be averaged in order to produce a conservation relationship for the deterministic components of the flow.



U = instantaneous vel
$\overline{\text{U}}$ = averaged over time
U' = deviations of one from the other

In turbulent flows velocity varies with time ⇒ ∴ we will characterize turbulent flow in terms of a mean or average velocity, its deterministic component

$$\underset{\sim}{u} = \overline{\underset{\sim}{u}} + \underset{\sim}{u}'$$

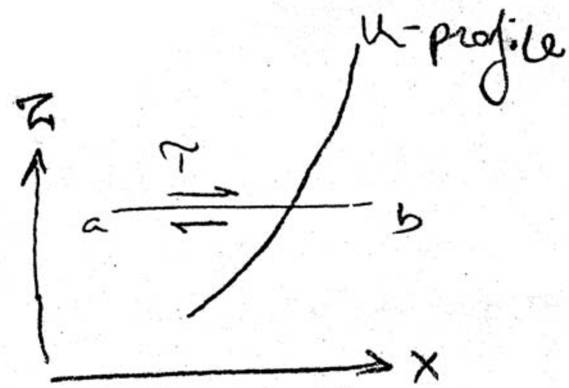
stochastic (fluctuating) component of the velocity term

deterministic portion of the turbulent flow velocity (CAN BE CALCULATED)

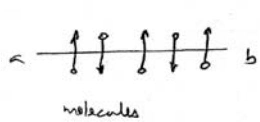
Early researchers on turbulence hypothesized turbulent eddies would have the same diffusive or mixing effect as molecular diffusion, although MUCH STRONGER.

<u>THE PROTOTYPE: Laminar flow</u>
Momentum transfer by molecular diffusion:

$$\tau_{zx} = \mu \frac{\partial u}{\partial z}$$



Even though the parcels of fluid are moving horizontally, molecular motion will transport momentum across surface a-b. Molecules above and below surface a-b are traveling at higher and lower velocities, respectively. The vertical motion of these molecules produces a resisting shear stress.

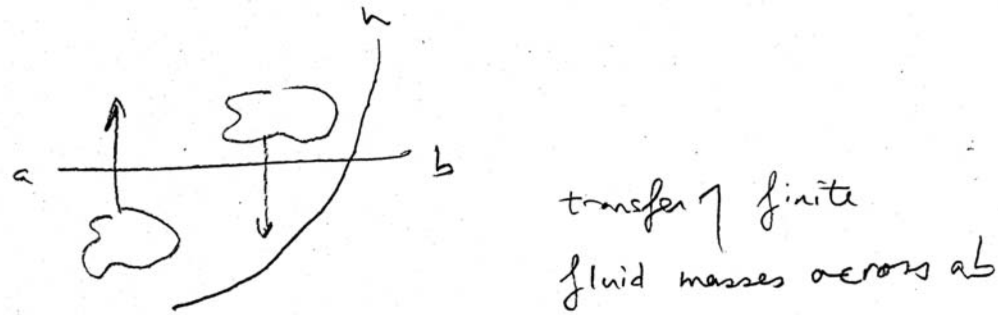


the molecules above & below line ab have faster & slower velocities, respectively.

produce a shear stress in opposite direction

THE MODEL: Turbulent flow

Momentum transfer by eddy transport:



Transfer of finite volumes of fluid across surface a-b redistributes fluid momentum, producing a resisting stress. This stress is commonly referred to as the ***Reynolds Stress***, τ_R.

Based on analogy to a laminar flow, they proposed an **Eddy Viscosity** closure for the description of the Reynolds stress/momentum flux.

$$\rho\overline{u'w'} \approx (\tau_{zx})_R = -\rho \nu_{turb}\frac{\partial \overline{u}}{\partial z} \qquad (\tau_{zx})_R = -\rho K \frac{\partial \overline{u}}{\partial z}$$

At large Reynolds Number: $\left|\tau_{reynolds}\right| >> \left|\tau_{viscous}\right|$

Important Definition

$$\tau_{zx_{total}} = \tau_b(1 - z/h)$$

Where

τ_b = boundary shear stress

z = distance above bed

h = flow depth

Constitutive relationship between Reynolds stress and mean strain rate

Assumptions:

1. Steady and horizontally uniform flow.
2. In a turbulent flow near a wall $\tau_R \cong \tau_b = \rho u_*^2$

stress **K** strain rate

LAW OF THE WALL

$$\tau_{zx} = \rho(\kappa u_* z)\frac{\partial u}{\partial z}$$

$$\tau_b(1 - z/h) = \rho(\kappa u_* z)\frac{\partial u}{\partial z}$$

$$\rho u_*^2(1 - z/h) = \rho \kappa u_* z \frac{\partial u}{\partial z}$$

small near bed

$$u_* = \kappa z \frac{\partial u}{\partial z}$$

Mixing length = κz with $\kappa = 0.407$, von Karman's constant of proportionality. This mixing length assumes that the dimension of turbulent eddies in the lower flow scale with distance from the boundary. Small eddies near the bed, larger eddies further away from the bed.

$$\int \frac{\partial u}{\partial z}\,dz = \int \frac{u_*}{\kappa z}$$

$$u = \frac{u_*}{\kappa}\int\frac{1}{z} = \frac{u_*}{\kappa}\ln z + \text{const}$$

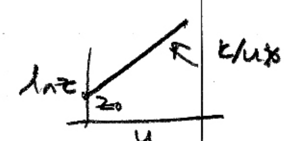
The Law of the Wall strictly applies to the flow near the bed ($z < 0.2h$). Empirically it provides a reasonable approximation for the entire velocity profile in most rivers.

$$0 = \frac{u_*}{\kappa}\ln z_0 + \text{const}$$

apply boundary condition at z_0, $\bar{u} = 0$

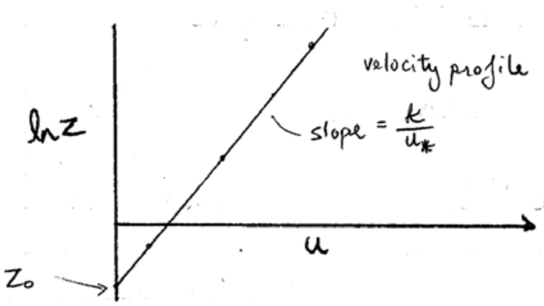
$$-\frac{u_*}{\kappa}\ln z_0 = \text{const}$$

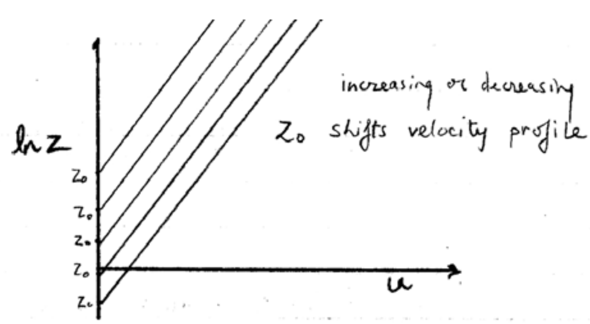
so, $$u = \frac{u_*}{\kappa}(\ln z - \ln z_0) = \frac{u_*}{\kappa}\left(\ln\frac{z}{z_0}\right)$$



The level z_0 is defined as the distance above the bed at which u = 0 if the turbulent velocity profile was extended downward to that position in the flow.

However, a viscous sublayer separates the turbulent flow from the bed. (An estimate for its thickness can be calculated taking the distance from the bed as the representative length scale for the Reynolds number.) It is therefore not valid to extrapolate the logarithmic velocity profile to $z = 0$.





Adjusting z_0 changes the value of u for a flow of constant u_*.

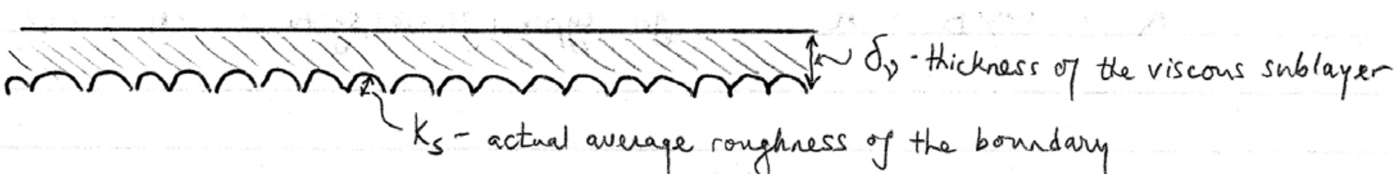
Measuring z_0: Boundary Roughness

Key to determining the appropriate roughness parameter is selecting the appropriate characteristic length scale for the bed roughness.

Selection of this scale is trivial in cases where the bed is composed of a single grain size. In this case the nominal diameter = $\boldsymbol{k_s}$, the roughness length scale.

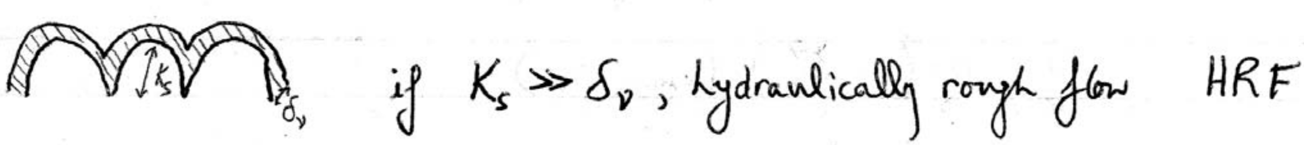
The trivial cases (One grain size, no sediment transport):

1. **Hydraulically Smooth Flow** [$k_s < \delta_\nu$, the average thickness of the viscous sublayer]



2. **Hydraulically Transitional Flow** [$k_s \approx \delta_\nu$, the average thickness of the viscous sublayer]

3. **Hydraulically Rough Flow** [$k_s > \delta_\nu$, the average thickness of the viscous sublayer]



Viscous sublayer effectively wraps around the roughness elements.

NOTE: The flow directly above δ_ν for the HRF case is accelerating and decelerating over the roughness elements. This interval of the flow does not satisfy the requirement of horizontally uniform flow assumed in derivation of the Law of the Wall. Quasi-uniform flow is set up at a distance about 3k above the bed.

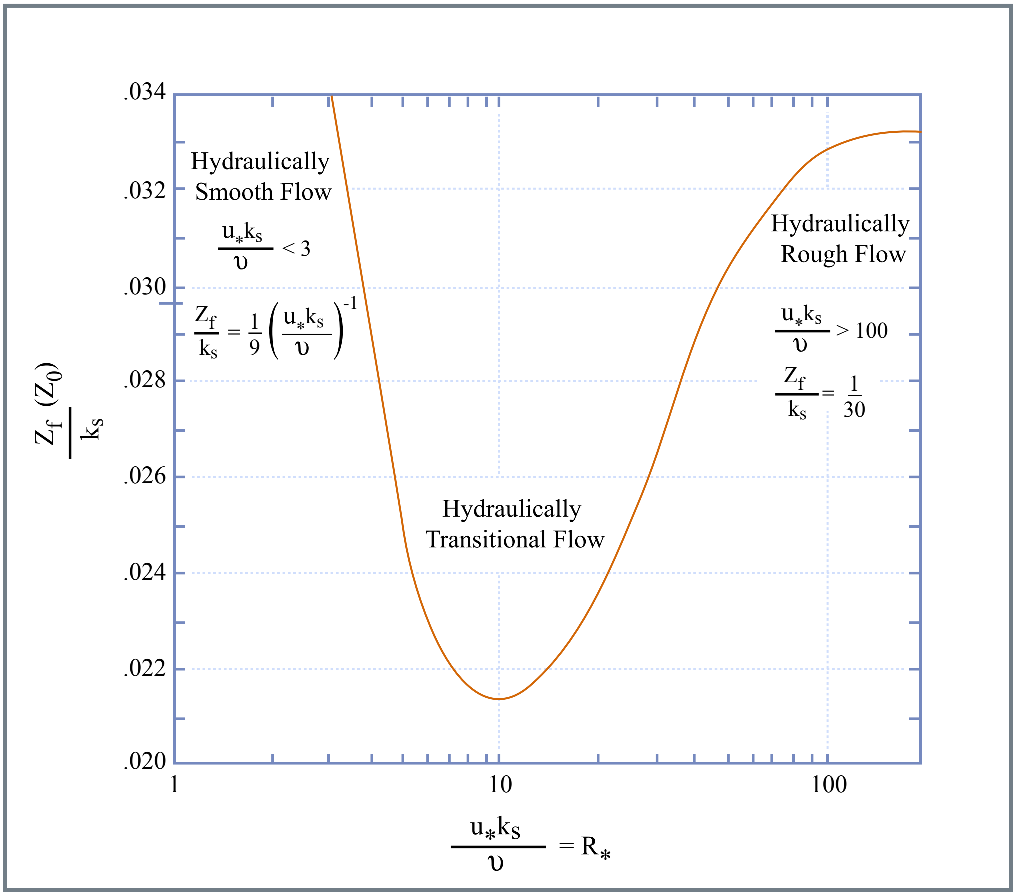


Figure by MIT OCW.

Nikuradse Diagram: $z_0 = k_s f(R_*)$ *where* $R_* = [u_* \times k_s]/\nu$
Nikuradse experimentally measured values for R_* as a function of z_0/k_s.

He did this by gluing well-sorted sand to the interiors of pipes and measuring pipe-flow velocity profiles.

Necessary first step. To attack most geological problems the relationship between needs to be expanded to handle 1) poorly sorted sediment (multiple potential roughness scales), and 2) bed irregularities (e.g., ripples, dunes, bars).