Week 1 notes: Stress, Mohr's circles

Fall 2005

1 Reading assignment

Chapters 1 and 2 in Twiss and Moores are introductory material, and well worth reading. If at any point during the course you feel like you aren't quite sure why we're talking about what we're talking about or you don't know where all this is going, you might want to flip back to these chapters.

Chapter 8 is required background for the lectures and labs on stress. Note that the notation used in the textbook is somewhat different from that used in the lectures.

2 Stress, tractions, forces

In geology, we seldom talk about forces. This is mainly because we're interested in the deformation in the interior of solid bodies. The relevant concept is **stress**:

$$\text{stress} = \frac{\text{force}}{\text{area}}$$

The units of stress are Newtons / meters^2, or Pascals. A force acting on a specified, particular area is known as a **traction**. However, we will often use **stress** and **traction** interchangeably. Like a force, a traction has a magnitude and a direction, and so is a vector.

We use two different vector notations. The simplest is to represent a vector by its components:

$$\vec{T} = [T_1\ T_2]$$

Alternatively, we can use direction cosines:

$$\vec{T} = \|T\|[\cos\alpha\ \ \cos\beta]$$

We are often concerned with distinguishing between the traction that acts normal to a surface (the **normal stress**) and the traction that acts parallel to the surface (the **shear stress**). Given some arbitrary plane and a traction $\vec{T}$ which makes the angle θ to its normal, the normal and shear stresses (tractions) are:

$$\sigma_N = \vec{T}\cos\theta$$
$$\tau = \vec{T}\sin\theta$$

3 Principal stresses

In continuous bodies, we analyze the stress on infinitesimal volumes of arbitrary orientations. We know the stress at a point in the body if we can determine the normal stress (traction) and shear stress (traction) that act on any plane passing through that point. The analysis of stress is often couched in terms of the tractions exerted on an infinitesimal cube whose faces line up with the coordinate axes.

We decompose the tractions exerted on any given face into one normal and 2 shear tractions. Using the standard naming convention, the normal tractions are: σ_{xx} σ_{yy} σ_{zz} and the shear tractions are: σ_{zy} σ_{zx} σ_{yx} σ_{yz} σ_{xy} σ_{xz}. These are typically arranged in a matrix:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Note This analysis makes the following assumptions.

1. The cube is infinitesimally small, no variations in traction for any face.

2. The cube is in mechanical equilibrium, tractions on opposite faces cancel.

3. No torques. That is: $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$,$\sigma_{yz} = \sigma_{zy}$ and there are only 6 independent components to the general 3-dimensional stress tensor.

The lack of torques, and therefore symmetry of the stress tensor implies that you can chose a coordinate frame such that all the shear stresses (the off-diagonal terms) vanish.

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \Longrightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

σ_1, σ_2, σ_3 are known as the principal stresses. The principal stresses are mutually perpendicular and are perpendicular to planes that suffer no shear stresses (the principal planes). If you have done any linear algebra, you may notice that the values of the principal stresses are just the eigenvalues; the direction of the principal stresses are the eigenvectors.

4 The stress ellipsoid

The stress ellipsoid is geometric representation of stress that allows us to visualize what the specific tractions would be on any given surface. The ellipsoid is defined as the envelope that contains the normal stresses to all planes of all possible orientations. Since its an ellipse, it is defined by three mutually perpendicular axes, which are the **principal stresses**.

The derivation of the equation follows from considering a small segment of an arbitrary plane **P**, with normal vector $\vec{P}$, subject to a traction $\vec{T}$. α, β, γ are the angles $\vec{P}$ makes with the **principal** coordinate axes. Let $\|\vec{P}\| = 1$ so that $\vec{P} = (\cos\alpha \;\; \cos\beta \;\; \cos\gamma)$ and $\|\vec{P}\| = \cos\alpha^2 + \cos\beta^2 + \cos\gamma^2 = 1$.

We begin by trying to solve for T_x, the component of the traction parallel to the x axis (which we have set to be parallel to one of the principal directions). We do this by solving a force balance. Since a stress is a force divided by an area, the force is the stress times the area. If A_P and A_x are the areas of P and the face perpendicular to the x direction, then

$$T_x A_P = \sigma_{xx} A_x$$
$$T_x = \sigma_{xx} \frac{A_p \cos\alpha}{A_p} = \sigma_{xx} \cos\gamma$$

Similarly, $T_y = \sigma_{yy} \cos\beta$ and $T_z = \sigma_{zz} \cos\gamma$ and

$$\cos\alpha = \frac{T_x}{\sigma_{xx}} \quad \cos\beta = \frac{T_y}{\sigma_{yy}} \quad \cos\alpha = \frac{T_z}{\sigma_{zz}}$$
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$
$$\frac{T_x^2}{\sigma_{xx}^2} + \frac{T_y^2}{\sigma_{yy}^2} + \frac{T_z^2}{\sigma_{zz}^2}$$

which is the equation of an ellipsoid.

5 Mohr circles

While the stress ellipsoid is technically a graphical representation of the state of stress, its actual practical utility is somewhat limited. Another representation is the Mohr circle construction, which isn't as intuitive or visual, but turns out to be more useful.

To derive the Mohr circle, we re-iterate that a stress is always physically associated with an area. It thus varies with the magnitude and orientation of the imposed force, but also with the area that you're interested in. For the derivation, we consider a rectangular block $ABCD$, whose top face is being squeezed such that a fracture of area = 1 runs through the block. We consider σ_1 to be normal to the top surface of the block and σ_3 to be normal to the sides of the block. Let θ be the angle *between the* **normal** *to the fracture and* σ_1, or the angle between the fracture and σ_3. We want to know the normal stress σ_N and the shear stress τ felt by the fracture surface.

Force balance: the **force** on AB is just σ_1 times the area of AB (=$\cos\theta$), so $F_{AB} = \sigma_1 \cos\theta$. The force on BC is $F_{BC} = \sigma_3 \sin\theta$. The force on the fracture is just $F_N = \sigma_N$. Now we're in a position to get an expression for σ_N: we balance F_N with the force acting on AB resolved on the fracture plus the force acting on BC resolved on the fracture.

$$\sigma_N = \sigma_1 \cos\theta \cos\theta + \sigma_3 \sin\theta \sin\theta$$
$$\sigma_N = \sigma_1 \cos^2\theta + \sigma_3 \sin^2\theta$$
$$\sigma_N = \frac{1}{2}\left[\sigma_1(1 + \cos 2\theta) + \sigma_3(1 - \cos 2\theta)\right]$$
$$= \frac{1}{2}(\sigma_1 + \sigma_3) + \frac{1}{2}(\sigma_1 - \sigma_3)\cos 2\theta$$

For the shear stress on the fracture:

$$\begin{aligned}\tau &= \sigma_1 \sin\theta\cos\theta - \sigma_3 \sin\theta\cos\theta \\ &= (\sigma_1 - \sigma_3)\sin\theta\cos\theta \\ &= \frac{1}{2}(\sigma_1 - \sigma_3)\sin 2\theta\end{aligned}$$

The equations we get are the parametric equations for a circle in σ vs. τ space.

Note that the above just considered the Mohr circle for 2 dimensions. We will discuss what the Mohr circle in 3 dimensions looks like in lab and class.

6 Special states of stress

We pay special attention to the following stress states:

1. Uniaxial stress
2. Biaxial and triaxial stress
3. Pure shear
4. Hydrostatic stress
5. Tension
6. Deviatoric stress