Week 2 notes: Strain

Fall 2005

1 Reading assignment

Twiss and Moores: chapter 15, in particular, pages 292 – 302. Discussion of specific special types of strain (pure shear, simple shear) begin on page 303. Pages 304 – 310 deal with progressive strain, and are useful background material for the lab.

W. Means (1976) *Stress and Strain* is a great text, very clear, well written and reads easily. J. Ramsay and M. Huber (1983) *The Techniques of Modern Structural Geology, Volume 1: Strain Analysis* is amazingly detailed, with many, many examples of detailed strain analysis. It can, however, be "a bit much".

2 Strain I: displacement, strain and terminology

Given enough stress, a material responds by deforming. We distinguish: between **rigid body deformations** and **non-rigid body deformations**. The first includes translation and rotations of a body. The second includes distortion and dilation. Other important distinctions are: **continuous** vs. **discontinuous** strain and **homogenous** vs. **heterogeneous** strain. Whether strain is homogenous or heterogenous is often a function of the scale of observation. Also, when strain in natural systems is analyzed, a common approach is to identify **structural domains** wherein the strain is continuous and sometimes homogenous. The point of doing so is that we can bring the tools of continuum mechanics – the physics of continuous deformation.

2.1 Measurement of strain

1. Changes in the lengths of lines

2. Changes in angles

3. Changes in areas or volumes

Changes in line length:

3 important measures:
Elongation

$$e \equiv \frac{\Delta l}{l_i} = \frac{l_f - l_i}{l_i} = \frac{l_f}{l_i} - 1$$

Stretch

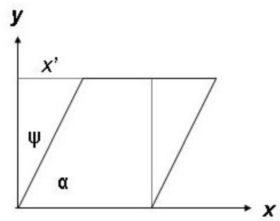
$$S \equiv \frac{l_f}{l_i} = 1 + e$$

Quadratic elongation

$$\lambda \equiv S^2 = (1+e)^2$$

So $\lambda = 1$ means no change in length; $\lambda < 1$ reflects shortening and $\lambda > 1$ is extension.

Changes in angles



1. Consider 2 originally perpendicular lines. The change in angle between those lines is

$$90 - \alpha = \psi \equiv \textbf{angular shear}$$

2. Consider a particle on the y-axis. Measure displacement at some distance y from the origin, in the x direction:

$$\frac{x}{y} = \gamma \equiv \textbf{shear strain}$$

Note that:

$$\gamma = \tan\psi$$

Change in volume (or area)

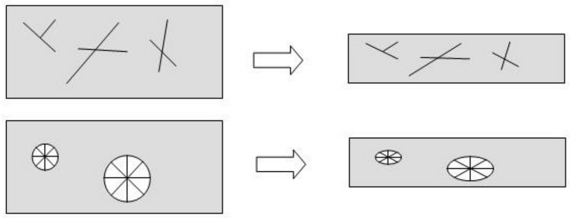
The **dilation** is similar to the definition of elongation:

$$\Delta \equiv \frac{V_f - V_i}{V_i} = \frac{\Delta V}{V_i}$$

Note that all these measurements take the undeformed state as the point of reference. It is equally feasible to take the length or angles in the deformed state as the reference state. There are no particularly good reasons for doing one or the other, neither is good for large strains. Alternatively, the **infinitesimal strain** is often a very useful concept, corresponding to the strain accrued in a vanishingly small instant of deformation.

3 The strain ellipsoid

In a deforming body, material lines will rotate and change shape. We want to be able to characterize the rotation and elongation of any arbitrary line. A circle (sphere in 3D) allows us to keep track of all possible orientations of lines. As it turns out, any homogenous strain turns a circle into an ellipse and a sphere into an ellipsoid.



Consider a circle of unit radius, deformed into an ellipse oriented such that the major and minor semi-axes are parallel with the coordinate axes. The elongation, stretch and quadratic elongation of the major semi-axis are given by:

$$e_x = \frac{l_{fx} - 1}{1}$$
$$S_x = 1 + e_x = l_{fx}$$
$$\lambda_x = l_{fx}^2$$

Alternatively, $l_{fx} = \sqrt{\lambda_x}$. Similarly, $l_{fz} = \sqrt{\lambda_z}$. Since the equation of an ellipse is just $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$, where a and b are the lengths of the semi-axes, the equation of the strain ellipse is just

$$\frac{x^2}{\lambda_x} + \frac{z^2}{\lambda_z} = 1$$

In three dimensions:

$$\frac{x^2}{\lambda_x} + \frac{y^2}{\lambda_z} + \frac{z^2}{\lambda_z} = 1$$

More generally, the semi-axes of the strain ellipse (ellipsoid in 3D) are the **principal strains**, analogous to the principal stresses we saw earlier. The length of the semi-axes (S_1, S_2, S_3) are the magnitudes of the principal strains. One of the features of the principal strains is that, not only are they orthogonal in the deformed state, but the same lines will be orthogonal in the undeformed state. The strain ellipsoid can have various shapes, corresponding to **uniaxial, biaxial** or **triaxial** strain. Circular sections through biaxial strain ellipses will be undistorted; in triaxial strain they will be distorted, though equally shortened or lengthened in all directions.

If you're lucky, the rock body you are looking at will contain initially circular or spherical markers. Deformation will turn these into ellipses, whose long and short axes are the principal strain axes. Examples of reasonably nice spherical strain markers are ooids or (perhaps) pebbles in a conglomerate. The spherical cross sections of worm tubes can also be used in this way. Initially elliptical or ellipsoidal markers can also be used to characterize strain, though this is more complicated.

4 Displacement vector fields and strain

One approach to analyzing strain is to keep track of particle displacements. The position (x, y) of a particle before deformation to its position (x', y') can be related by a set of coordinate transformation equations of the form:

$$x' = ax + by$$
$$y' = cx + dy$$

If a, b, c, d are constants, then the strain is homogenous. Two particularly important strain regimes are **simple shear** and **pure shear** and their coordinate transformation equations, expressed in matrix notation are:

$$\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} k & 0 \\ 0 & 1/k \end{bmatrix}$$

Deformation is usually not instantaneous: strains accumulate over time. One kind of strain can follow one another. Mathematically, this is equivalent to multiplying the strain matrices. Note that matrix multiplication is not commutative, so, for example, simple shear followed by pure shear does not yield the same final result as pure shear followed by simple shear.

If the orientations of the principal strain axes do not rotate during deformation, then the strain is said to be **irrotational**.

5 Mohr circles for strain I : Infinitesimal strain

The **infinitesimal strain** is a useful concept that can be thought of as representing the instantaneous material response to stress. The accumulation of infinitesimal strain increments over geological time results in the deformation that the geologist observes and tries to understand in the outcrop – known as the **finite strain**. For our purposes we can consider "pretty small" strains – say, less than 1% – to be infinitesimal. For the purposes of deriving the Mohr circle equations, this case also allows the use of small angle approximations, in particular, $\gamma = \psi$.

As with stress, we want a Mohr circle construction that allows us to read off (1) the elongation (for infinitesimal strain, call it ϵ) and (2) the shear strain (γ that affect any line of any given orientation. The principal strains are denoted $\epsilon_1, \epsilon_2, \epsilon_3$. A fairly tedious derivation (for details, you might check out Hobbes (1976)) yields:

$$\epsilon = \left(\frac{\epsilon_1 + \epsilon_2}{2}\right) + \left(\frac{\epsilon_1 - \epsilon_2}{2}\right)\cos 2\alpha$$
$$\frac{\gamma}{2} = \left(\frac{\epsilon_1 - \epsilon_2}{2}\right)\sin 2\alpha$$

Note that this Mohr circle is drawn in ϵ vs. $\gamma/2$ coordinates. Examination of the Mohr circle for infinitesimal strain yields the following important relations:

1. There are two lines that experience the maximum shear strain, and they are located at 45°to the principal strain axes.
2. The maximum shear strain is given by $\gamma/2 \pm (\epsilon_1 - \epsilon_2)/2$, i.e. $\gamma \pm (\epsilon_1 - \epsilon_2)$.
3. Any 2 lines perpendicular to one another are 180°apart on the Mohr circle, so they suffer shear strains equal in magnitude but opposite in sign.

6 Mohr circles II: Finite strain

As the result of finite strain, lines are lengthened or shortened and the angles between intersecting lines are usually changed. Considering a unit circle deformed into an ellipse whose axes are parallel with the coordinate frame, we can derive relationships that track the elongations and rotations (shear strains) for any line. As you might expect, a Mohr circle construction is the ticket to the big time. The derivations of these are so tedious that even Ramsay and Huber relegate them to an appendix (cf. Ramsay and Huber, appendix D if you can't help yourself... I myself have wasted a greater part of my mortal existence on these than I would really care to admit ....). Two separate constructions are available: one identifies lines according to the angles they make with the principal strain directions in the "unstrained state" – this is of somewhat limited use since we rarely know what the orientations of lines used to be. The other deals with the orientations in the strained state.

6.1 Unstrained state reference frame

The unstrained reference frame refers back to the pre-deformation orientation of a line P. The line makes an angle θ with the principal strain axes. Upon deformation, it suffers an elongation and rotates into a new position P' (in general, we will use primes to distinguish the reference frames) making an angle θ' with the principal strain directions. The elongation and shear strain of an arbitrary line are given by

$$\lambda = \frac{(\lambda_1 + \lambda_2)}{2} + \frac{(\lambda_1 - \lambda_2)}{2} \cos 2\theta$$
$$\gamma = \frac{\lambda_1 - \lambda_2}{2\sqrt{\lambda_1 \lambda_2}} \sin 2\theta$$

Note: these fail to produce a circle (instead, you get a Mohr *ellipse*) unless $\sqrt{\lambda_1 \lambda_2} = 1$, i.e. there is no dilation.

6.2 Strained reference frame

Usually, we are confronted with good rocks gone bad, and so the angles we measure are those of lines in the strained state, i.e. we measure θ', not θ. Conversion between the two reference frames can be done using:

$$\sin \theta = \lambda^{1/2} \sin \theta' / \lambda_2^{1/2}$$
$$\cos \theta = \lambda^{1/2} \cos \theta' / \lambda_1^{1/2}$$

and pulling a little definitional slight of hand:

1. Define a **new strain parameter $\gamma' = \gamma/\lambda$.**
2. Use the **reciprocal quadratic extensions** $\gamma'_1 = 1/\gamma_1$ and $\gamma'_2 = 1/\gamma_2$.

The Mohr equations become:

$$\lambda' = \frac{\lambda'_1 + \lambda'_2}{2} - \frac{\lambda'_1 - \lambda'_2}{2} \cos 2\theta'$$
$$\gamma' = \frac{\lambda'_2 - \lambda'_1}{2} \sin 2\theta'$$