Notes IV
Rheology part 1: Ideal material behaviours

Fall 2005

1 Reading assignment

Chapter 18 in Twiss and Moores, pages 361 – 385, deal with both ideal models for rock deformation as well as experimental investigation of ductile flow. Chapter 9, pages 165 – 190 deal with the mechanics of brittle fracture. This is good stuff, believe.

2 Ideal behaviours

The next section of the course will review what we know about the relationship between stress and strain; that is, the laws that govern deformation. We start with "ideal" behaviours, and then compare these to experimental results. A material's response to an imposed stress is called the **rheology**. A mathematical representation of the rheology is called a **constitutive law**.

2.1 Elastic behaviour – Hooke's law

The characteristics of elasticity are: 1. strain is **instantaneous** upon application of stress; 2. stress and strain are **linearly** related; 3. strain is perfectly **recoverable**.

$$\sigma \propto \epsilon$$

There ought to be a constant of proportionality on there, called the **elastic modulus**; the exact modulus you use depends on whether the strain is volumetric, uniaxial or in shear. Respectively, these are:

$$\sigma_h = K\epsilon_v$$
$$\sigma_n = E\epsilon_n$$
$$\sigma_s = 2\mu\epsilon_s$$

Where K, E, μ are the bulk modulus, Young's modulus and the shear modulus. Another important elastic constant is ν, Poisson's ratio. Poisson's ratio relates the elastic strain for orthogonal directions.

2.2 Viscous behaviour – Newtonian fluids

Strain in a viscous fluid is **time dependant** and **non-recoverable**. Time-dependance is the fundamental difference with elastic behaviour: instead of a linear relationship between stress and strain, viscous materials exhibit a linear relationship between stress and **strain-rate**.

$$\tau = \eta\dot{\epsilon}$$

The constant of proportionality, η, is the **viscosity**, and has units of Pascal-seconds. A plot of strain against time is a line, provided that the viscosity remains constant. Viscosity is a measure of the strength of a material: higher viscosity materials are stronger. Typical geological viscosities are:

Material	Viscosity (Pa-s)
Water	10^{-3}
Lava	0.1 - 10
Glacier ice	10^{13}
Salt	10^{14} - 10^{20}
Window glass	10^{21}
Athenospheric mantle	10^{21}

The effective viscosity can change during the course of deformation. If the viscosity increases, we call the process **strain hardening** or **shear rate thickening**; conversely, if the material becomes weaker and the viscosity decreases, the process is **strain softening** or **shear rate thinning**.

2.3 Visco-elasticity I: Maxwell bodies

If we represent elastic behaviour with a spring, and viscous behaviour with a dashpot[1], we can also combine dashpots and springs in a number of different ways to simulate various possible ideal material responses or rheologies.

The most basic combination is to put a spring and a dashpot in series. This combination is known as **Maxwell viscoelasticity**. The constitutive relationship for Maxwell viscoelasticity comes from the linear addition of the relationships for elasticity and viscous fluids[2]:

$$\dot{\epsilon} = \frac{\dot{\sigma}}{2\mu_M} + \frac{\sigma}{2\eta_M}$$

The subscript M just indicates that we're dealing with Maxwell-rigidity and Maxwell-viscosity. This equation can be manipulated by stipulating either constant stress ($\sigma = \sigma_0$) or constant strain ($\dot{\epsilon} = 0$). Under constant stress:

$$\epsilon = \frac{\sigma_0}{2\mu_M} + \frac{\sigma_0}{2\eta_M}t$$

[1] What the heck is a dashpot? They show up a lot in material science, geodynamics and differential equations, and are invariably introduced as though they are a familiar, intuitive part of one's day to day experience. Except that they aren't, really. Ask the random person on the street what a dashpot is and you'll get a pretty blank look. Dashpots are simple pistons combined – usually – with a hydraulic fluid. Examples of objects that include dashpots are car shocks, bike shocks, and are found on some doors.

[2] The notation I use differs somewhat from what Clark used in class. Mine follows standard texts such as Ranalli, G. (1995) *Rheology of the Earth*, and Turcotte, D. and Schubert (2002) *Geodynamics*.

On a strain vs. time plot, this shows instantaneous elastic strain, followed by steady state linear viscous strain. The case of constant strain is a little more difficult to intuit, but the equation, and plot of stress against time are clear enough.

$$\sigma = \sigma_0 \exp\left(-\frac{\mu_M}{\eta_M} t\right)$$

A plot of stress against time demonstrates the viscous stress relaxation of a Maxwell body. That is, a Maxwell body should relax to an isotropic (i.e. hydrostatic) stress state on a time scale that is captured by the ratio of $\eta_M : \mu_M$. Expressed as a fraction, this ratio has units of time, and is known as the **Maxwell time**. The Maxwell time of the asthenospheric mantle is on the order of a thousand years and is what sets the time scale of phenomena like post glacial rebound.

2.4 Visco-elasticity II: Kelvin bodies

Another way to combine a spring and a dashpot (that is, an elastic response and a viscous response) are to put them in parallel, rather than in series. This material is known as a **Kelvin** body, and sometimes called **fermoviscous** behaviour. This is the idealization of a phenomenon (that is, something that actually happens in the real world) called **elastic afterworking**, which is just that, in the real world, most springs and supposedly elastic materials don't always respond instantaneously to the imposed stress. A Kelvin body shows **time-dependant**, **recoverable** strain. Kelvin behaviour is given by

$$\sigma = 2\mu_K \epsilon + 2\eta_K \dot{\epsilon}$$

Upon loading, the elastic response of the spring is damped by the dashpot, and goes asymptotically to $\sigma_0 / 2\mu_K$:

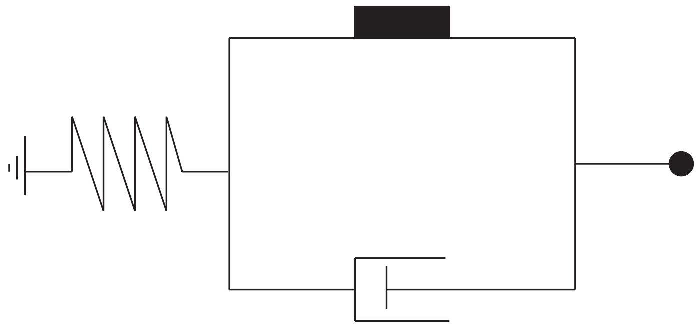
$$\epsilon = \frac{\sigma_0}{2\mu_K}\left[1 - \exp\left(-\frac{\mu_K}{\eta_K} t\right)\right]$$

When the load is removed, the strain is recovered, but not instantaneously. Suppose, a strain ϵ_0 had been accummulated, then $\epsilon(t)$ is

$$\epsilon = \epsilon_0 \exp\left(-\frac{\mu_K}{\eta_K} t\right)$$

2.5 Other ideal rheologies

Four other ideal behaviours are worth mentioning. The first are two end-members of Newtonian viscosity: the **Pascal liquid** and the **Euclid solid**, which have viscosities 0 and ∞, respectively. The last two are types of **plastic** behaviour. A plastic material is one that has a yield stress, but otherwise behaves as a Newtonian fluid (or a non-Newtonian, power-law fluid). The **St. Venant** body (also called an **elastic-plastic** body) is idealized as a friction block being pulled by an elastic. Such a body is a material that exhibits linear elasticity up to a certain point, the yield point. Then, the material fails abruptly. A **Bingham** body, also called **viscoplastic**, is idealized as a spring in parallel with a friction



block *itself in parallel* with a dashpot[3]. Its behaviour is governed by the equations:

$$\sigma = 2\mu\epsilon \quad \sigma < \sigma_Y$$
$$\sigma = \sigma_Y + 2\eta_B\dot{\epsilon} \quad \sigma \geq \sigma_Y$$

The Bingham body behaves elastically at stresses lower than the yield stress, and flows as a linear fluid above the yield strength with a strain rate proportional to $\sigma - \sigma_Y$. Examples of Bingham materials are certain clays, some kinds of submarine debris flows, oil paintings, drilling muds, toothpaste and bread dough.

3 Study and review questions

You should be aware of the differences between: linear elastic, linear viscous, Maxwell viscoelastic, and Kelvin behaviour. St. Venant and Bingham bodies are also relevant since they introduce the concept of a yield stress.
For each of the above, draw a spring and dashpot idealization, and draw curves that relate stress and strain (or strain-rate). For Maxwell and Kelvin bodies, also plot stress against time for constant strain, and strain (or strain rate) against time for constant stress. Think off a real world example or two for each behaviour.

[3]Note that in class, this was shown, in error, as a dashpot and a friction block in series. Twiss and Moores leave out the leading spring, which is a special case.