12.510 Introduction to Seismology
Spring 2008

12.510 Lecture Notes
3.3.2007

The Wave Equation

$\ddot{\phi} = \alpha^2 \nabla^2 \phi$, for a P wave

- Often written $\alpha^2 \nabla^2 \phi - \ddot{\phi} = 0$ or $L(\phi) = 0$ where L is an operator.
- Using d'Alembert's Solution: $\phi(\bar{x}, t) = A(\bar{x}) e^{i(\bar{k} \cdot \bar{x} - \omega t)}$, where the wave number $\bar{k}$ indicates the direction of the wave

Ray Parameter and Slowness

A useful way to characterize a wave's ray path is via its slowness, the reciprocal of the apparent velocity.

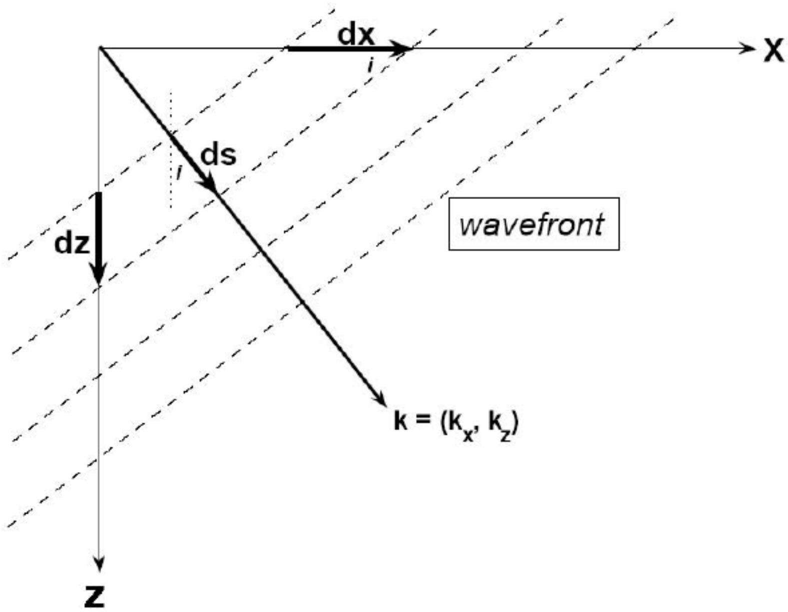


Figure 1. The arrow is used for a ray and the dashed line is used for a wavefront. The wavenumber $\bar{k}$ indicates the direction of the ray. The angle i is both the take-off angle and the angle of incidence.

We define the wave speed, c = ds/dt, with horizontal wave speed, c_x = dx/dt, and vertical wave speed, c_z = dz/dt. Using Figure 1 we can relate the angle if incidence with the horizontal and vertical wave speed.

$$\sin(i) = \frac{ds}{dx} = c\frac{dt}{dx} = \frac{c}{c_x} \equiv cp$$

$$\cos(i) = \frac{ds}{dz} = c\frac{dt}{dz} = \frac{c}{c_z} \equiv c\eta$$

Here p is horizontal slowness, also known as the ray parameter, and η is vertical slowness.

$$p \equiv \frac{1}{c_x} = \frac{\sin(i)}{c} \qquad \eta \equiv \frac{1}{c_z} = \frac{\cos(i)}{c}$$

The slowness vector, $\bar{s} = (p, \eta)$, is composed of the horizontal and vertical slowness. Some properties of the slowness vector:

$$s = \sqrt{p^2 + \eta^2} = \frac{1}{c} \qquad p^2 + \eta^2 = \frac{1}{c^2}$$

However, the addition of squares of horizontal wave speed and vertical wave speed does not equal to squares of wave speed, $c_x^2 + c_z^2 \neq c^2$. In addition, we will examine critical phenomenon in reflection and refraction with the relation $\eta = \sqrt{\frac{1}{c^2} - p^2}$. In terms of wave number, each component of wave number can be represented by horizontal and vertical slowness.

$$k_x = \frac{\omega}{c_x} = \omega p \qquad\qquad k_z = \frac{\omega}{c_z} = \omega\eta$$

Thus, wave number speed is related to the slowness vector.

$$\bar{k} = (k_x, k_z) = (\omega p, \omega\eta) = \omega(p, \eta) = \omega\bar{s}$$

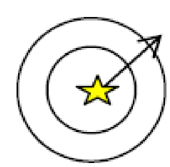
Geometric Ray Theory

Remember from plane wave superposition:

$$\phi(\bar{x}, t) \approx \int_{-\pi}^{\pi}\int\int_{-\infty}^{\infty} A(\ldots)e^{i(\bar{k}\cdot\bar{x}-\omega t)}dk_x dk_y d\omega$$

We will use a high frequency approximation, the limit as $\omega \to \infty$, which leads to geometric ray theory. We can gain insight into the behavior of the seismic waves by considering the ray paths associated with them. This approach, studying wave propagation using ray path, is called geometric ray theory. Although it does not fully describe important aspects of wave propagation, it is widely used because it often greatly simplifies the analysis and gives a good approximation.

A wave front connects points of equal phase: equal phase ~ equal travel time, T, from the origin.



Eikonal Equation

- *eikon* = image (Greek)

Consider the following solution to the wave equation, $\ddot{\phi} = \alpha^2\nabla^2\phi$:

$$\phi(\bar{x}, t) = A(\bar{x})e^{i(\bar{k}\cdot\bar{x}-\omega t)}$$

We choose to work at a travel time, $T(\bar{x})$.

$$\phi(\bar{x}, t) = A(\bar{x})e^{-i\omega T(x)}$$

Working to insert this expression back into the wave equation:

$$\nabla\phi = \nabla A e^{-i\omega T} - i\omega A\nabla T e^{-i\omega T}$$

$$\nabla^2\phi = \left(\nabla^2 A - \omega^2 A|\nabla T|^2 - i(2\omega\nabla A\cdot\nabla T + \omega A\nabla^2 T)\right) e^{-i\omega T}$$

$$\ddot{\phi} = -\omega^2 A e^{-i\omega T}$$

$$\Rightarrow \underbrace{\nabla^2 A - \omega^2 A|\nabla T|^2}_{\text{Real}} \underbrace{- i\omega(2\nabla A\cdot\nabla T + A\nabla^2 T)}_{\text{Imaginary}} = -\frac{A\omega^2}{\alpha^2} = -Ak_\alpha^2$$

note: $k_\alpha = \frac{\omega}{\alpha}$, $k_\beta = \frac{\omega}{\beta}$ and for the general case $k = \frac{\omega}{c}$

For information on propagation, consider just the real part.

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$$\nabla^2 A - \omega^2 A |\nabla T|^2 = -\frac{A\omega^2}{\alpha^2}$$

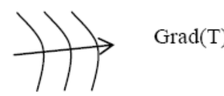
$$|\nabla T|^2 - \frac{1}{\alpha^2} = -\frac{\nabla^2 A}{A\omega^2}$$

Apply the high frequency approximation and take the limit as $\omega \to \infty$. For sufficiently large ω the right-hand side goes to 0.

$$|\nabla T|^2 = \frac{1}{\alpha^2} \text{ for a P wave}$$

For the general case:

$$|\nabla T|^2 = \frac{1}{c^2} \Rightarrow |\nabla T(\bar{x})| = \frac{1}{c(\bar{x})}$$



$$\nabla T(x) = \frac{1}{c(x)}\bar{k} = \bar{s} = (p, \eta) = \text{slowness vector}$$

What does it mean?

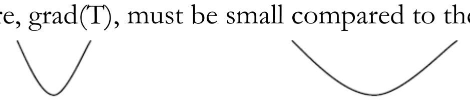
Gradient of a wavefront at a position x (here defined as the travel time, surface of equal phase) is equal to the local slowness. The direction of maximum change of the wavefront defines the direction of the wave propagation.

What are the implications?

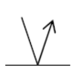
- Rays are perpendicular to wavefronts.
- The slowness gives the gradient of the travel time, and the gradient of the travel time specifies the direction of the ray. Each time c(x) changes, the gradient of T has to change, and the direction of propagation changes at the same time.
- If one knows c(x), there is a way to reconstruct the direction of the ray: eikonal ray tracing.

Warning: ω needs to be sufficiently large, but it does not need to be infinite for the eikonal equation to be a valid simplification of the wave equation. There are no fixed rules but some conditions of validity exist:

- Change in wave speed along the ray has to be small i.e. the distance over which C(x) changes has to be large compared to the wavelength.
- Curvature, grad(T), must be small compared to the wavelength.

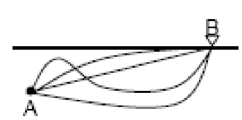


Small curvature   Large curvature



Extreme case: reflection or infinite curvature. This can be studied as long as you consider infinitely high frequencies → infinitely narrow rays.

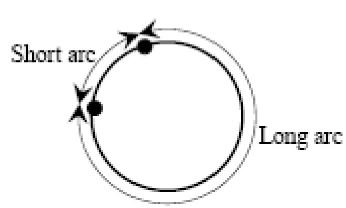
Fermat's Principle



Consider the kinetic and potential energy along an arbitrary path between two points, A and B. Stationary points of the integrated difference between KE and PE over all possible paths specify paths of "least action".

There are an infinite number of paths from A to B, but there is only B one 'correct' path: the one with the shortest travel time.

Both the shortest and the longest paths are stationary points.



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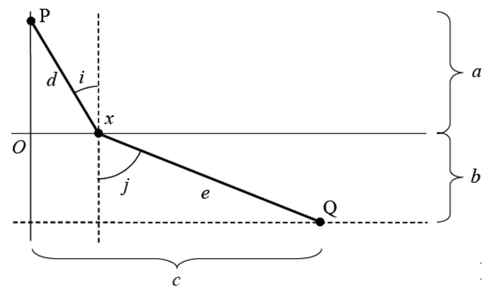
Consider the mathematical formulation of the problem.

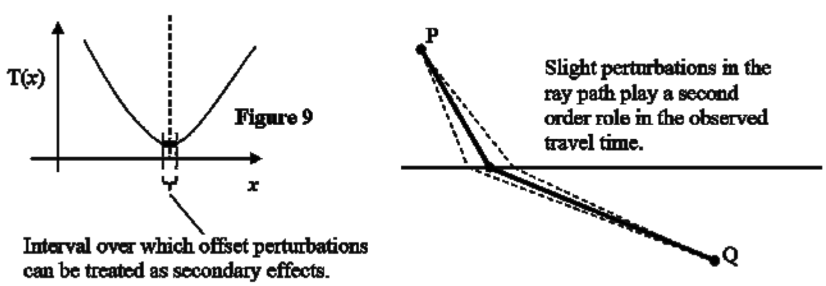


$$T_{P-Q} = \frac{d}{c_1} + \frac{e}{c_2}$$
$$= \frac{\left(a^2 + x^2\right)^{1/2}}{c_1} + \frac{\left(b^2 + (c-x)^2\right)^{1/2}}{c_2}$$

$$\frac{dT}{dx} = \frac{x}{c_1\left(a^2 + x^2\right)^{1/2}} + \frac{c - x}{c_2\left(b^2 + (c-x)^2\right)^{1/2}} = \frac{\sin i}{c_1} - \frac{\sin j}{c_2} = 0 \quad (\textit{optimum}!)$$

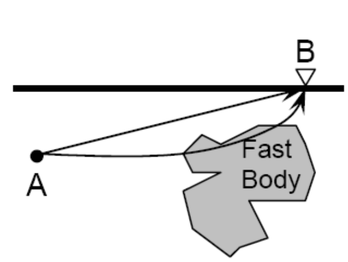
Hence, $\frac{\sin i}{c_1} = \frac{\sin j}{c_2}$. Fermat's Principle implies Snell's Law.

The travel time curve, plotted as a function of offset, is typically a hyperbolic function. Near stationary points, the curve is usually fairly 'flat', which implies that, near optimum or stationary points, the travel time is locally insensitive to slight variations in offset. Consequently, close to a stationary point, small deviations in the ray path can be treated as second order effects.



For example…

- The presence of a fast body embedded in a homogeneous medium causes the reference (optimal) ray path to deflect from its original path between A and B.
- But, as a consequence of Fermat's Principle, the change in ray path produces a second order effect on the arrival time.
- The effect on travel time of changes in wave speed (elastic parameters) is first order.



Implication: We can generate reference ray paths assuming a homogeneous medium and use this reference model to simplify subsequent analysis for heterogeneous media. → Important for travel time tomography.

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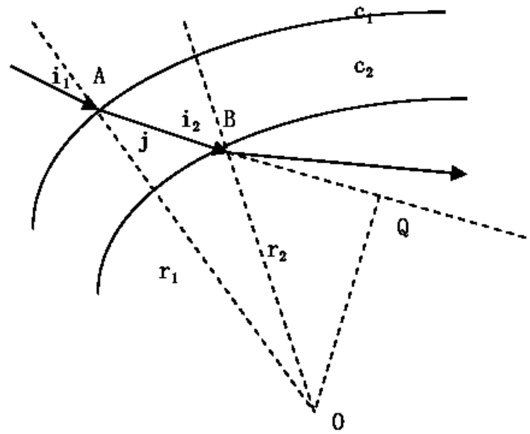
Snell's Law in a Spherical Medium



At each interface

$$\frac{\sin i_1}{c_1} = \frac{\sin j}{c_2}$$

$$\sin j = \frac{OQ}{OA} \quad \sin i_2 = \frac{OQ}{OB}$$

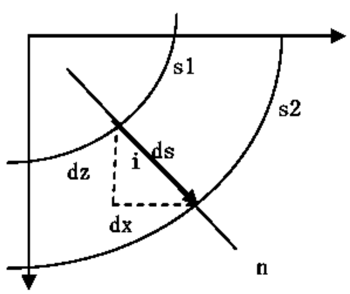
$$\sin j = \frac{OB}{OA}\sin i_2 = \frac{r_2}{r_1}\sin i_2$$

$$\boxed{\frac{r_1 \sin i_1}{c_1} = \frac{r_2 \sin i_2}{c_2} \equiv p}$$

"flat earth" → $p = \frac{\sin i}{c}$ "spherical earth" → $p = \frac{r \sin i}{c}$

At critical angle, $p = \frac{r_p}{c(r_p)}$ we can get depth of layer.

Ray Equation



Directional cosine (3D and 2D)

$$(\frac{dx_1}{ds})^2 + (\frac{dx_2}{ds})^2 + (\frac{dx_3}{ds})^2 = 1 \qquad (\frac{dx}{ds})^2 + (\frac{dz}{ds})^2 = 1$$

Direction of ray ($\hat{n}$)

$$\hat{n} = (n_x, 0, n_z) \qquad n_x = \frac{dx}{ds} \qquad n_z = \frac{dz}{ds}$$

Using Eikonal equation $\nabla T = \frac{1}{c}\hat{n}$,

Generalized Snell's law (Ray Equation)

$$\boxed{\frac{d}{dx_i}(\frac{1}{c(x)}) = \frac{d}{ds}(\frac{1}{c(x)}\frac{dx_i}{ds})}$$

This equation means that the change of wavespeed is related to change of ray geometry.
If there is no change in x direction, the derivative of x direction should be zero.

$$\frac{d}{ds}(\frac{1}{c(x)}\frac{dx}{ds}) = 0 \quad \Rightarrow \quad \frac{1}{c}\frac{dx}{ds} = Const. \quad \Rightarrow \quad \frac{\sin i}{C} = Const. \quad \Rightarrow \textit{Snell's law !!}$$

How does this angle i change in the direction of propagation?

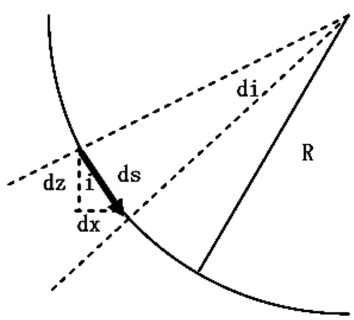
$$\frac{d}{ds}(\sin i) = \cos i \frac{di}{ds} = \frac{dz}{ds}\frac{di}{ds} = \frac{d}{ds}(pc) \qquad \Rightarrow \qquad \boxed{\frac{di(s)}{ds} = p\frac{dc}{dz}}$$

Therefore, the change of angle is related to the change of velocity.

If $\frac{dc}{dz}$ is large $\Rightarrow$ $\frac{di}{ds}$ is large

If $\frac{dc}{dz}$ is zero (c = const.) $\Rightarrow$ $\frac{di}{ds}$ is zero (i = const.) *Straight Ray !!*

Radius of Curvature



R : the radius of curvature

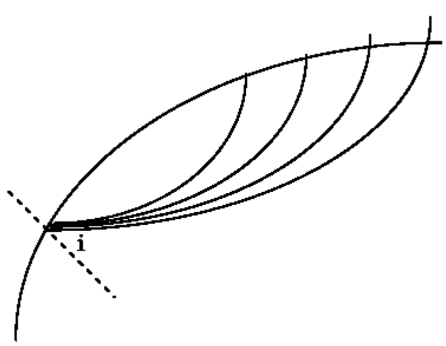
$ds = Rdi$

$$R = \frac{ds}{di} = \frac{1}{p}\frac{dz}{dc} = \frac{1}{p(\frac{dc}{dz})} \quad \Rightarrow \quad \boxed{R = \frac{1}{p(\frac{dc}{dz})}}$$

R is related to wavespeed gradient and ray parameter.

If $\frac{dc}{dz} = 0$ $\Rightarrow R \to \infty$ *Straight Ray !!*

If $\frac{dc}{dz}$ large $\Rightarrow$ rapid change in c *Strong Gradient*



from $p = \frac{r \sin i}{c}$,

small i → small p → large R