12.520 Lecture Notes 27

Flow in Porous Media

Problem of great economic importance (also scientific)

- hydrology (ground water migration, toxic waste)
- oil migration
- soil stability, fault mechanics (pore pressure)
- melt migration in mantle
- geysers and hot springs

Porous medium $\Rightarrow$ voids $\Rightarrow$ porosity ϕ

$\phi \equiv$ volume fraction of voids

For example,

Sand: $\phi \sim 40\%$

Pumice: $\phi \sim 70\%$

Oil shales: $\phi \sim 10\text{–}20\%$

If pore connected $\Rightarrow$ permeable

Pressure gradient $\Rightarrow$ flow

Darcy's law $\Rightarrow \underset{\sim}{v} = -\dfrac{k}{\eta} \nabla p$

$v \equiv$ volumetric flow rate $k \equiv$ permeability

We can use Poiseuille flow for simple geometries. For example, cubical matrix, circular tubes or pipes.

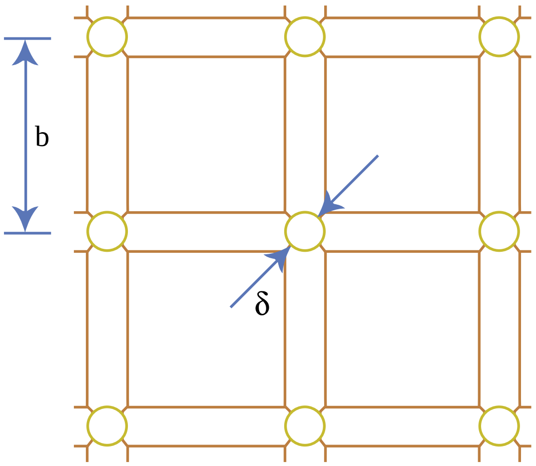

Figure 27.1. An idealized model of a porous medium. Circular tubes of diameter δ form a cubical matrix with dimensions b.

Figure 27.1

Figure by MIT OCW.

$$\phi = \frac{12 \cdot \frac{1}{4} \cdot \pi \cdot \left(\frac{\delta}{2}\right)^2 \cdot b}{b^3} = \frac{3\pi}{4}\frac{\delta^2}{b^2}$$

Consider $\frac{dp}{dx}$ (one direction only)

In each pipe (along x), $\bar{u} = -\frac{\delta^2}{32\eta}\frac{dp}{dx}$ [Poiseuille flow]

Darcy velocity: $v = \dfrac{4 \cdot \frac{1}{4} \cdot \bar{u} \cdot \pi \cdot \left(\frac{\delta}{2}\right)^2}{b^2} = \dfrac{\pi\delta^2}{4b^2}\bar{u} = \dfrac{\phi}{3}\bar{u}$

$$v = -\frac{b^2\phi^2}{72\pi\eta}\frac{dp}{dx}$$

$$\Rightarrow \quad k = \frac{1}{72\pi}b^2\phi^2$$

Large $b \Rightarrow$ large v? $\quad b^2 = \frac{3\pi}{4}\frac{\delta^2}{\phi}$

Large $\phi \Rightarrow$ large v? $\quad k = \frac{\pi}{128}\frac{\delta^4}{b^2}$

Compare to cubes separated along faces (channel flow)

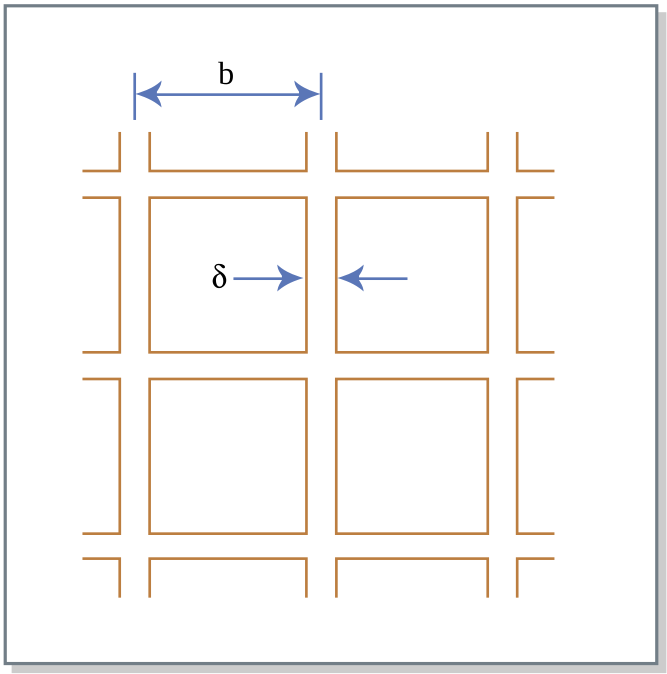

Figure 27.2

Figure by MIT OCW.

$$\phi = \frac{6 \cdot \frac{1}{2} \cdot \delta b^2}{b^3} = 3\frac{\delta}{b}$$

Again, $\frac{dp}{dx}$ directed along one edge

$$u = \frac{1}{2\eta}\frac{dp}{dx}\left(Z^2 - (\delta/2)^2\right)$$

$$\bar{u} = \frac{1}{2\eta\delta}\frac{dp}{dx}\left(\frac{Z^3}{3} - \frac{\delta^2 Z}{2}\right)\Bigg|_{-\delta/2}^{\delta/2} = -\frac{5\delta^2}{24\eta}\frac{dp}{dx}$$

Darcy velocity: $v = 2\frac{b\delta}{b^2}\bar{u} = -\frac{5}{12}\frac{\delta^3}{b\eta}\frac{dp}{dx} = -\frac{5}{324}\frac{b^2\phi^3}{\eta}\frac{dp}{dx}$

$$k = \frac{5b^2\phi^3}{324}$$

k is different depending on ϕ.

$$k = \frac{135}{324}\frac{\delta^3}{b}$$

Clearly, porosity distribution is important.

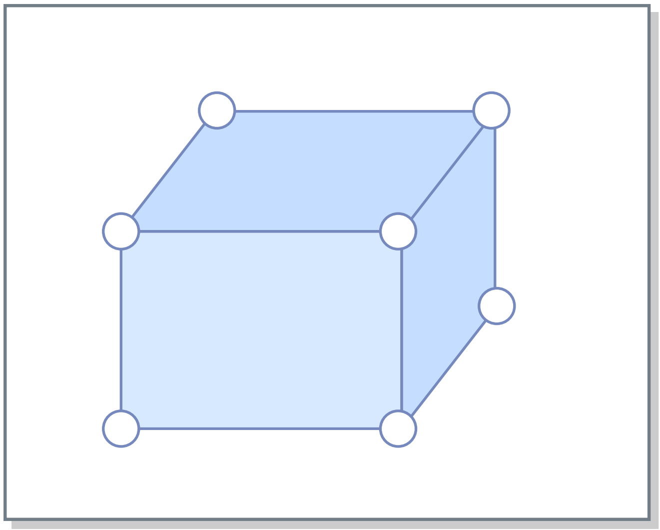

Figure 27.3

Figure by MIT OCW.

Also -- more easily measured than figured out theoretically – more complicated geometries → numerical simulation.

Consider “Lawn Sprinkler” example – flow in unconfined aquifer.

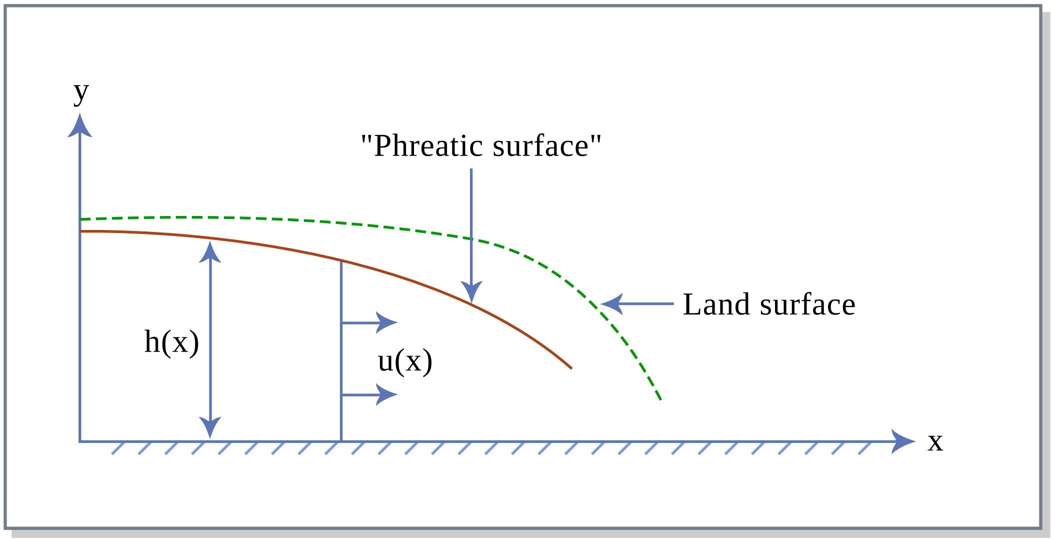

Figure 27.4
Figure by MIT OCW.

$h \equiv$ "hydraulic head"

$u \rightarrow$ Darcy velocity

Dupuit approximation: $\frac{dp}{dx} = \rho g \frac{\partial h}{\partial x}$

For $\frac{\partial h}{\partial x} = 1$ flow is one-dimensional.

Darcy's law: $u = -\frac{k\rho g}{\eta}\frac{\partial h}{\partial x}$

Conservation of mass: Assume no input

Flux $Q = u(x)h(x) = -\frac{k\rho g}{\eta} h \frac{dh}{dx} = \text{const.}$

$\Rightarrow$ phreatic surface is a parabola

For $h = h_0$ at $x = 0$

$$h = \left(h_0^{\ 2} - \frac{2Q\eta x}{k\rho g}\right)^{1/2}$$

Suppose we have a porous dam of width *w*. The relation between Q, h_0 and h_1 is:

$$Q = \frac{k\rho g}{2\eta w}\left(h_0^2 - h_1^2\right)$$

or

$$Q = \frac{k\rho g}{2\eta w}\left[\left(h_0 - h_1\right)\left(h_0 + h_1\right)\right]$$

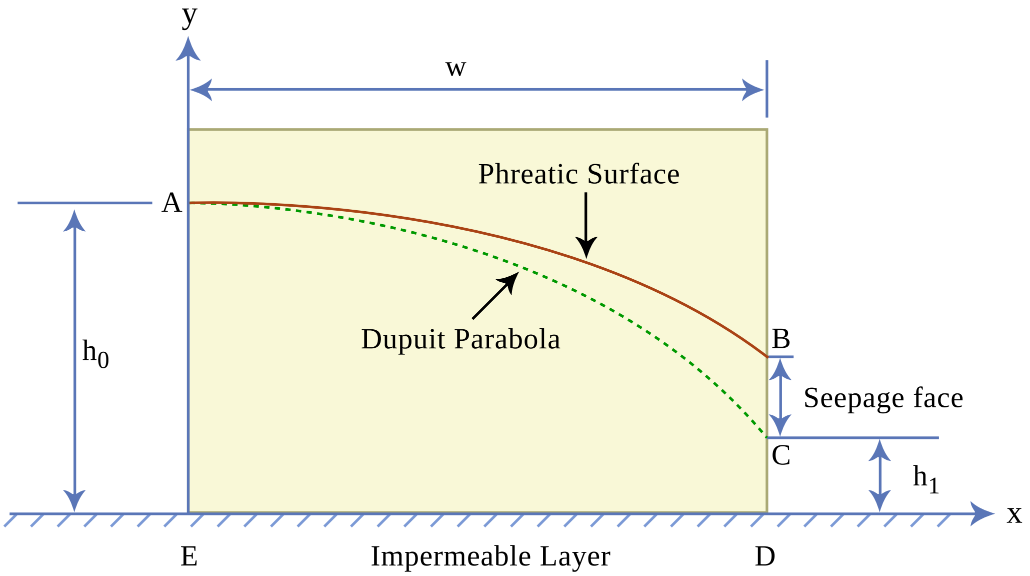

Figure 27.5. Unconfined flow through a porous dam. The Dupuit parabola AC is the solution if $(h_0-h_1)/h_0 \ll 1$. The actual phreatic surface AB lies above the Dupuit parabola resulting in a seepage face BC.

Figure 27.5

Figure by MIT OCW.