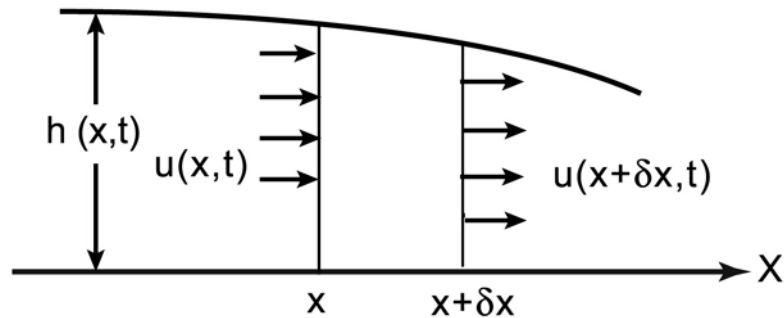


12.520 Lecture Notes 28

Time Dependent Porous Flow

Assume 1-D flow through an unconfined aquifer.



$$\text{Flux in } Q_{in} = u(x,t)h(x,t)$$

$$\text{Flux out } Q_{out} = u(x + \delta x, t)h(x + \delta x, t)$$

$$Q_{out} - Q_{in} \approx \frac{\partial}{\partial x}(uh)\delta x \Rightarrow \frac{\partial h}{\partial t}$$

But medium is porous, with porosity ϕ . For a given ΔQ , the smaller ϕ , the larger $\partial h / \partial t$

$$Q_{out} - Q_{in} = -\phi[h(t + \delta t, x) - h(t, x)]\delta t \approx \phi \frac{\partial h}{\partial t} \delta x \delta t$$

$$\text{Mass conservation} \Rightarrow \phi \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

$$\text{Darcy's law (Dupuit approx)} \Rightarrow u = -\frac{k\rho g}{\eta} \frac{\partial h}{\partial x}$$

$$\text{Combining} \Rightarrow \frac{\partial h}{\partial t} = \frac{k\rho g}{\eta\phi} \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) \quad (\text{"Boussinesq Equation"})$$

This is a nonlinear diffusion equation. We can simplify it by assuming

$$h = h_0 + h' \quad h' = h_0$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) &= \frac{\partial}{\partial x} \left[(h_0 + h') \frac{\partial}{\partial x} (h_0 + h') \right] \\ &= \frac{\partial}{\partial x} \left[(h_0 + h') \frac{\partial h'}{\partial x} \right] \\ &\approx h_0 \frac{\partial^2 h'}{\partial x^2} \end{aligned}$$

This gives the “familiar” diffusion equation

$$\frac{\partial h'}{\partial t} = \frac{k \rho g h_0}{\eta \phi} \frac{\partial^2 h'}{\partial x^2}$$

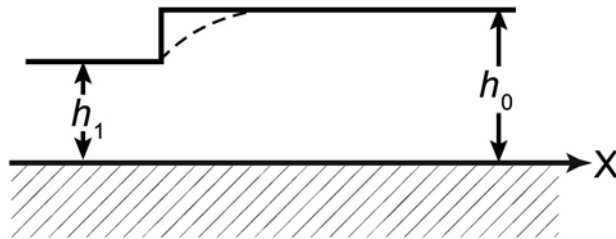
$\frac{k \rho g h_0}{\eta \phi}$ equivalent to K in heat flow equation

equivalent to $\nu = \eta / \rho$ in fluid $\frac{1}{2}$ space

[note inverse relationship of fluid viscosity η !]

Consider the sudden lowering of water in a river channel next to a saturated bank.

$$h(x^+, 0) = h_0 \quad h(0, t) = h_1 \quad h(\infty, t) = h_0$$



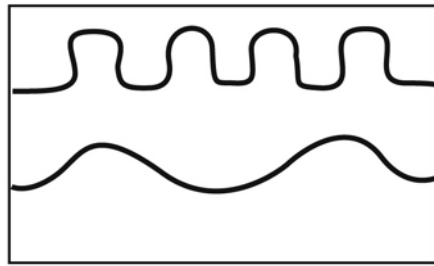
Solution is like the solution to the heat flow equation for plate cooling, or the momentum equation for crew shell:

$$\text{Set } f = \frac{h}{h_0} \quad f(0) = \frac{h_1}{h_0}$$

$$\xi = \left(\frac{\eta \phi}{k \rho g h_0 t} \right)^{1/2} \frac{x}{2}$$

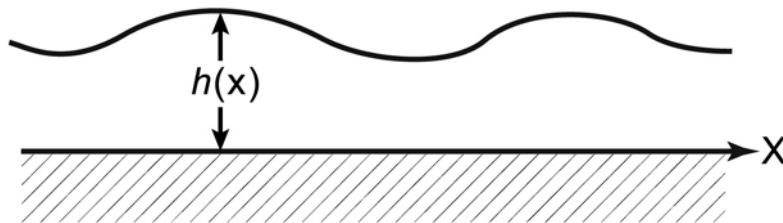
$$f(\xi) = f(0) \operatorname{erfc}(\xi)$$

Finally, back to the tank demo

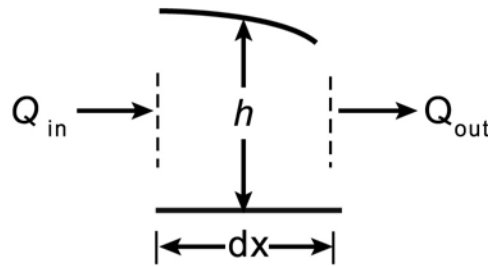


Flow between glass plates \approx Darcy flow.
 (recall 1 model of porosity)

Analyze a simple case-1 boundary



To conserve fluid



$$Q_{out} - Q_{in} = -dx \cdot \frac{dh}{dt} \cdot \phi$$

$$\phi \frac{\partial h}{\partial t} = -\frac{dQ}{dx}$$

but $Q = u \cdot h$, with $u \propto \frac{dh}{dx}$

$$\phi \frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}(uh) = \frac{k\rho g}{\eta} \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) \leftarrow \text{nonlinear!}$$

Linearize around $h = h_0 + h_1$

$$\frac{\partial h'}{\partial t} = \frac{k\rho gh_0}{\phi\eta} \frac{\partial^2 h'}{\partial x^2} \quad \leftarrow \text{diffusion equation}$$

\neq exponential growth or decay

[Note: Dupuit approximation breaks down for large interface perturbation.]

Suppose $h' = \rho \cos kx$, $k = 2\pi / \lambda = \text{wave number}$

$$\frac{\partial^2 h'}{\partial x^2} = -k^2 \rho \cos kx$$

$$\frac{\partial \rho}{\partial t} = -\frac{K\rho gh_0}{\phi\eta} k^2 \rho \quad \text{where } K \text{ is permeability.}$$

Exponential growth or decay!

$$\tau = \frac{\eta\phi}{\rho gh_0 K} \cdot \frac{1}{k^2}$$

How does this compare to Rayleigh-Taylor?