12.520 Lecture Notes 12

Elasticity

So far:

Stress $\rightarrow$ angle of repose vs accretionary wedge

Strain $\rightarrow$ reaction to stress $\rightarrow$ but how?

Constitutive relations

$$\tau_{ij} = \tau_{ij}\left(\varepsilon_{kl}\right) \; ; \quad \varepsilon_{ij} = \varepsilon_{ij}\left(\tau_{kl}\right)$$

For example,

- Elasticity
  - Isotropic
  - Anisotropic
- Viscous flow
  - Isotropic
  - Anisotropic
- Power law creep
- Viscoelasticity

Trade offs:

simplicity	$\leftrightarrow$	**realism**
constant		variable
isotropic		anisotropic
elastic, viscous		viscoelastic
history independent		history dependent

Tensors

Most physical quantities that are important in continuum mechanics like temperature, force, and stress can be represented by a tensor. Temperature can be specified by stating a single numerical value called a scalar and is called a zeroth-order tensor. A force, however, must be specified by stating both a magnitude and direction. It is an example of a first-order tensor. Specifying a stress is even more complicated and requires stating a magnitude and two directions—the direction of a force vector and the direction of the normal vector to the plane on which the force acts. Stresses are represented by second-order tensors.

Tensors are quantities independent of coordinate system.

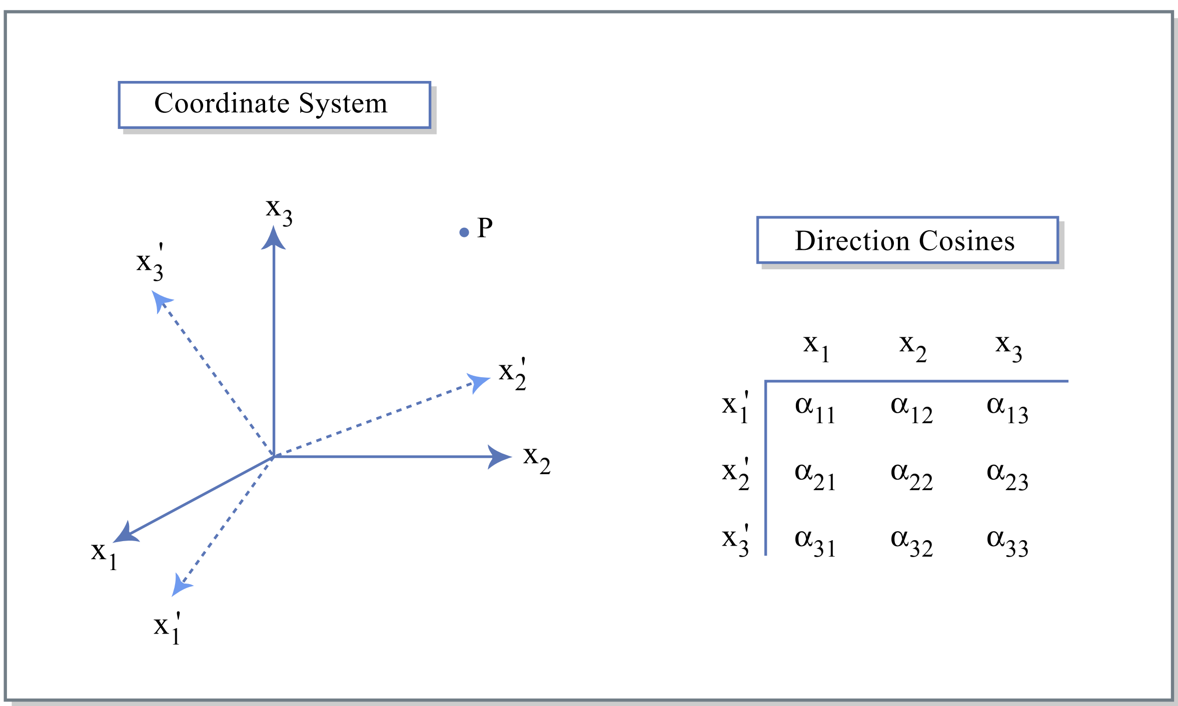


Figure 12.1

Figure by MIT OCW.

$$\alpha_{ij} = \cos\phi_{ij}$$

where ϕ_{ij} is the angle of primed to original.

$$x_i' = \alpha_{ij} x_j$$

$$x_i = \alpha_{ji} x_j'$$

$$\alpha_{ij} = \frac{\partial x_i'}{\partial x_j} = \frac{\partial x_j}{\partial x_i'}$$

Tensors:

a. 0[th] order (scalar) – quantity dependent only on position
b. 1[st] order (3^1 components) $A_i' = \alpha_{ij} A_j$
c. 2[nd] order ($3^2 = 9$ components) $A_{ij}' = \alpha_{is}\alpha_{jk} A_{sk}$
d. 3[rd] order ($3^3 = 27$ components) $A_{ijk}' = \alpha_{is}\alpha_{jt}\alpha_{kp} A_{stp}$
e. 4[th] order ($3^4 = 81$ components) $A_{ijkl}' = \alpha_{is}\alpha_{jt}\alpha_{kp}\alpha_{lq} A_{stpq}$

Conventional moduli:

1. Hydrostatic comp.

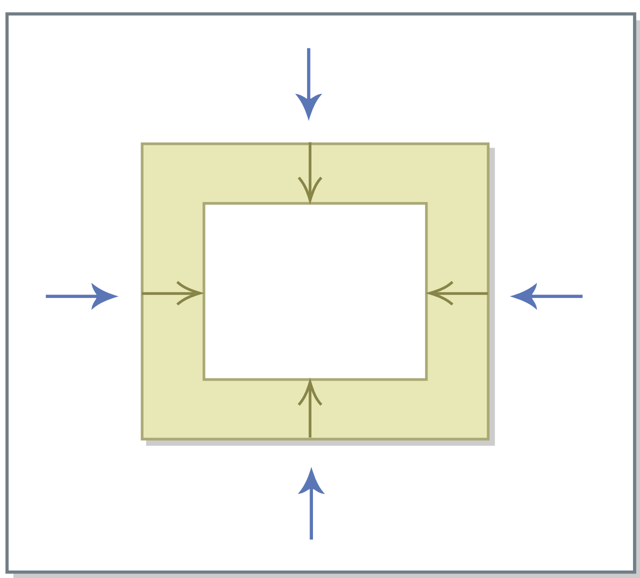


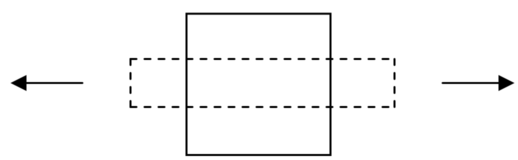
Figure 12.2

Figure by MIT OCW.

$$\tau_{ij} = -p\delta_{ij}$$
$$\tau_{ii} = -p\delta_{ii} = 3\lambda e_{kk} + 2\mu e_{ii}$$
$$= -3p = (3\lambda + 2\mu)e_{ii}$$
$$-\frac{p}{e_{ii}} = -\frac{VP}{\Delta V} \equiv K$$

where $K = \lambda + 2/3\mu$ is bulk modulus.

2. Uniaxial stress



$$\tau_{11} = T$$
other $\tau_{ij} = 0$
$$2\mu e_1 = T - \frac{\lambda}{2\mu + 3\lambda} T$$
$$\frac{T}{e_1} \equiv E \text{ (sometimes } Y\text{)}$$

where $E = \dfrac{\mu(2\mu + 3\lambda)}{\mu + \lambda}$ is Young's modulus

Hook's law:

$$T = Ee$$
$$\frac{e_{22}}{e_{11}} = \frac{e_{33}}{e_{11}} \equiv -\nu$$ This is called Poisson's ratio.
$$2\mu e_{22} = -\frac{\lambda}{2\mu + 3\lambda}\tau_{11} \quad \Rightarrow \quad \nu = \frac{\lambda}{2(\mu + \lambda)}$$
$$\theta = e_{11} + e_{22} + e_{33} = e_{11}(1 - 2\nu)$$
fluid: $\mu \to 0 \quad \Rightarrow \quad \nu \to \frac{1}{2}$

most material: $\nu = 0.2 - 0.3$

$\nu = \frac{1}{4} \quad \Rightarrow \quad \lambda = \mu$ It is Poisson solid.

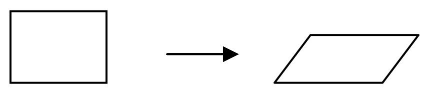
steel: ν ; $0.3-0.33$

seismically measured

$$v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad v_s = \sqrt{\frac{\mu}{\rho}}$$

compare $v_p, v_s \rightarrow \nu \rightarrow$ discriminate rock types

3. Simple shear



$$\tau_{12} = \tau_{21} = \tau$$
$$\tau_{12} = 2\mu e_{12} = 2Ge_{12}$$

where G is shear modulus.

Note: Among λ, μ, K, ν, E, G only two are independent.