3. Surface gravity waves

The most familiar form of wave motions are the waves occurring at the interface between the atmosphere and water, like the waves we see on the beach. The restoring force of these waves is gravity, hence they are called surface gravity waves

Let us consider a homogeneous layer of fluid with a free surface at z = $\eta(x,y,t)$ and a constant depth –D. We want to ignore rotation, friction and non-linearity: can we? Scale analysis.

(a) to ignore rotation compare $\frac{\partial}{\partial t} = \frac{1}{T} = \omega$ with Ω. This implies $\omega >> \Omega$ or $T << f^{-1}$. The period of the motion is small enough to ignore rotation in the equation of motion

$$\frac{d\vec{u}}{dt} + 2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho_o}\underline{\nabla} p + \mu \nabla^2 \vec{u} = gk$$

(b) To ignore friction compare $\mu \nabla^2 \vec{u}$ with $\frac{\partial \vec{u}}{\partial t}$. This is

$$\mu \nabla^2 \vec{u} = \mu \frac{u}{\lambda^2} = \mu u k^2$$

with k a typical wave number and $\frac{\partial \vec{u}}{\partial t} = u\omega$

This means $\omega >> \mu k^2$

(c) To ignore nonlinearity compare $\frac{\partial \vec{u}}{\partial t}$ with $\vec{u} \bullet \underline{\nabla} \vec{u}$ or $\frac{1}{T} = \omega$ with $\frac{u^2}{\lambda} = u^2 k$

$\omega >> uk$

or $u << \frac{\omega}{k} = c$

the particle speed is much less than the phase speed, i.e. the signal is carried by the wave and not by the advective motion.

(d) Then we want to treat the fluid as incompressible. Suppose the motion is adiabatic. Then entropy is conserved following a fluid particle. s is the specific entropy and the general equation is $\frac{ds}{dt} = H$

where H are heat sources/sinks. If the motion is adiabatic H = O.

s = s(p,ρ). Assume we can linearize:

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} = 0 = \frac{\partial s}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial s}{\partial \rho}\frac{\partial \rho}{\partial t}$$

thus $\frac{dp}{dt} = -\frac{\partial s/\partial \rho}{\partial s/\partial p}\frac{d\rho}{dt} = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{\partial \rho}{\partial t}$

The speed of sound in any medium is given by the adiabatic compressibility of the medium, i.e. by $\left(\frac{\partial p}{\partial \rho}\right)_s$.

$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ c_s = sound speed ≡ adiabatic compressibility of the medium.

So the relationship between pressure and density perturbation is:

$\delta p = c_s^2 \delta\rho$ $\rho = \bar{\rho} + \delta\rho \simeq \bar{\rho}$ in momentum e.g. as $\delta\rho << \bar{\rho}$

From horizontal momentum equation:

$$\nabla p = 0\left(\bar{\rho}\frac{\partial u}{\partial t}\right)$$

or $\frac{\delta p}{\lambda} = 0\left(\frac{\bar{\rho} u}{T}\right) \rightarrow \delta p k = 0(\bar{\rho} u \omega)$

$$\delta p = 0\left(\frac{\bar{\rho} u \omega}{k}\right)$$

From $\delta \rho = c_s^2 \delta \rho$

$$\delta \rho = \frac{\delta \rho}{c_s^2} = 0\left(\frac{\bar{\rho} u \omega^2}{k c_s^2}\right)$$

and $\dfrac{\delta(\delta \rho)}{\partial t} = 0\left(\dfrac{\bar{\rho} u \omega^2}{k c_s^2}\right)$

Mass conservation

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \bullet (\rho \vec{u}) = 0$$

usual convention $\rho = \bar{\rho} + \delta \rho$; $\bar{\rho}$ = grand average = constant

$$\frac{\partial(\partial \rho)}{\partial t} + \underline{\nabla} \bullet \left[(\bar{\rho} + \delta \rho)\vec{u}\right] = 0 \qquad \delta \rho << \bar{\rho}$$

$$\frac{\partial(\partial \rho)}{\partial t} + \bar{\rho} \bullet \underline{\nabla} \bullet \vec{u} = 0 \qquad \text{->} \qquad \frac{1}{\bar{\rho}} \frac{\partial(\partial \rho)}{\partial t} + \underline{\nabla} \bullet \vec{u} = 0$$

But $\dfrac{1}{\bar{\rho}} \dfrac{\partial(\partial \rho)}{\partial t} = 0\left(\dfrac{u \omega^2}{k c_s^2}\right)$ and $\nabla \bullet \vec{u} = 0(uk)$

So $\dfrac{1}{\bar{\rho}} \dfrac{\partial(\partial \rho)}{\partial t} / \underline{\nabla \bullet} \vec{u} = 0 \left(\dfrac{\omega^2}{k c_s^2 k}\right) = 0\left(\dfrac{\omega^2 / k^2}{c_s^2}\right) = 0\left(\dfrac{c^2}{c_s^2}\right)$

c = phase speed = 10 to 100 m/sec

c_s = sound speed = 1,500 m/sec in the ocean

$\dfrac{1}{\bar{\rho}} \dfrac{\partial(\delta \rho)}{\partial t} / \underline{\nabla} \bullet \vec{u} << 1 \;\Rightarrow\; \underline{\nabla} \bullet \vec{u} = 0$ => incompressibility

If $c \ll c_s$ then we can consider the fluid as incompressible; c_s = 1,500 m/sec in the ocean (Not in the atmosphere: $c_s \simeq$ 300 m/sec, of the order of the phase speed of internal waves). Mass conservation equation for incompressible flow,

$$\underline{\nabla} \bullet \vec{u} = 0$$

The equations of motion is (keeping nonlinearity for the moment)

$$\frac{d\vec{u}}{dt} = -\frac{1}{\rho_o}\underline{\nabla}p - g\hat{k} \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u} \bullet \underline{\nabla}$$

Vorticity is defined as $\vec{\zeta} = \underline{\nabla} \times \vec{u}$

Take the curl of the momentum equation and linearize:

$$\frac{d}{dt}(\nabla \times \vec{u}) = 0 \quad \Rightarrow \frac{d\vec{\zeta}}{dt} = 0 \quad \Rightarrow \frac{\partial\vec{\zeta}}{\partial t} = 0$$

So if the relative vorticity is zero at initial time (or at any other time) it will be zero at all times. In general this gives $\vec{\zeta}$ = constant at all times: the constant is arbitrary $\equiv$ 0.

If $\vec{\zeta} = \underline{\nabla} \times \vec{u} = 0$, we can define a velocity potential ϕ such that $\vec{u}(x,t) = \underline{\nabla}\phi(\vec{x},t)$

Since the fluid is incompressible, $\underline{\nabla} \bullet \vec{u} = 0$ and

$$\nabla^2\phi = 0 \qquad\qquad u=\phi_x,\ v=\phi_y,\ w=\phi_z$$

is the equation for the velocity potential, an elliptic problem very simple equation describing, among other things, the electric potential of static charges. The dynamics of surface waves is contained in the boundary conditions.

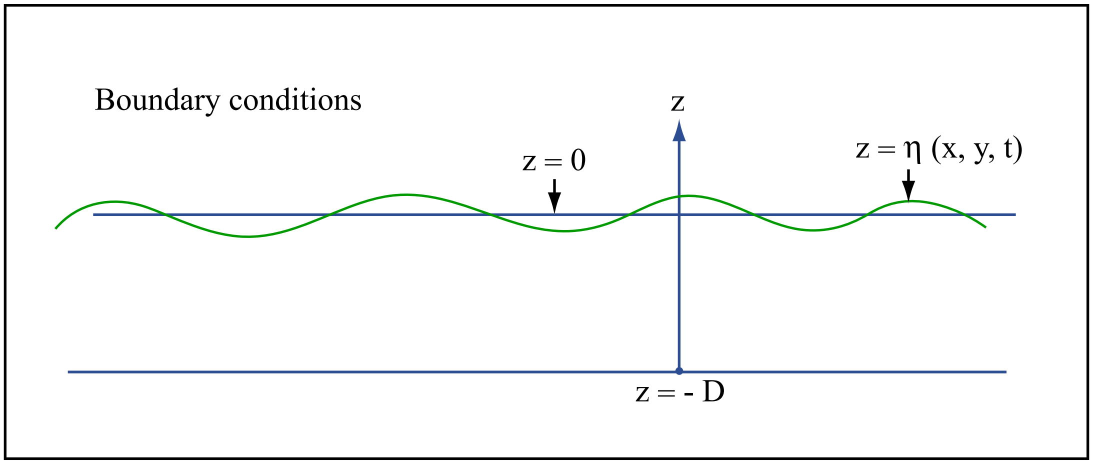

Figure by MIT OpenCourseWare.

Figure 1.

We now derive the boundary conditions keeping nonlinearity, and we shall linearize later. At the bottom z = -D we require w = O, i.e.

$$\phi_z = 0 \quad \text{at} \quad z = -D$$

The free surface η is made of fluid parcels that move with the fluid velocity field which never leave the interface. Consider one such parcel. It moves vertically (i) if the interface rises or falls, or (ii) if the fluid flows horizontally under the sloping interface. If we let z = η(x,y,t) be the interface, then

$$w[x,y,\eta\,(x,y,t),t] = \eta_t + u\eta_x + v\eta_y \quad \text{at} \quad z = \eta$$

This is really just a restatement of Dη/Dt = w. In terms of φ, this says

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z \qquad \text{at} \quad z = \eta$$

This is nothing more than as kinematic condition which simply says what we mean by calling z = η an interface.

The interface is massless. In the absence of surface tension, therefore it supports no pressure differences across it. The appropriate dynamical boundary condition is

$$p(x,y,\eta,t) = p_{atmosphere}$$

To write this in terms of φ, η return to

$$\vec{u}_t + (\vec{u} \bullet \nabla)\vec{u} = -\underline{\nabla} p/\rho_o - g\hat{k}$$

Using the identity

$$(\overline{u} \bullet \underline{\nabla})\vec{u} = (\underline{\nabla} \times \vec{u} + \underline{\nabla}(\vec{u} \bullet \vec{u}/2)$$

we can rewrite this (exactly) as

$$\vec{u}_t + \vec{\zeta} \times \vec{u} = -\underline{\nabla} p/\rho_o - \underline{\nabla}(\vec{u} \bullet \vec{u}/2) - \underline{\nabla}(gz)$$

Now if $\overline{\zeta} = 0$ so that $\vec{u} = \underline{\nabla}\phi$, then this becomes

$$\underline{\nabla}(\phi_t + p/\rho_o + \frac{1}{2}|\nabla\phi|^2 + gz) = 0$$

$$\text{or} \qquad \phi_t + gz + \frac{1}{2}|\nabla\phi|^2 = f(t) - \frac{p}{\rho_o}$$

which is the Bernoulli integral. We apply this at $z = \eta$ where $p = p_{atm}$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = f(t) - p_{atm}/\rho$$

We can always add a g(t) to ϕ without changing its physical meaning: $\nabla^2\phi = 0$ Add a g(t) to ϕ such as $\frac{\partial g}{\partial t} = f(t)$. In this way we can eliminate f(t).

We consider free waves i.e. not forced. Then $p_{atm} = 0$ and f(t) = arbitrary = 0.

Notice how a specified $p_{atm}(x,y,t)$ would enter the problem through this boundary condition.

The full problem is

$$\frac{d\eta}{dt} = \eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z \quad \text{at} \quad z = \eta$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0 \quad \text{at} \quad x = \eta$$

$$\nabla^2\phi = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -D$$

3.2 Linear solutions

To get some idea of possible solutions, we will linearize and solve in one horizontal dimension. For now we just drop the nonlinear terms. We will check a posteriori that they are small compared with the linear terms. The linearized problem is:

$$\eta_t = \phi_z \quad \text{at} \quad z = 0$$

$$\phi_t + g\eta = 0 \quad \text{at} \quad z = 0$$

$$\nabla^2\phi = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -D$$

What sets the amplitude of the motion is the amplitude of the free surface a, which is small. Linearization, as previously discussed, implies neglecting all nonlinearities. As every dynamic variable is of the order of the amplitude of the motion a, this implies neglecting terms of $0(a^2)$, i.e. nonlinear, quadratic terms $\rightarrow$ a must be small. Now take any of the terms in the two boundary condition equations, and call it $G(x,y,\eta)$ as they are applied at $z = \eta$. Expand it around $\eta = 0$:

$$G(x,y,\eta) = G(x,y,o) + \eta \frac{\partial G}{\partial \eta}\Big|_{z=o} + \text{higher order terms}$$

The first term $G(x,y,0) = 0(a)$. The second term is of $0(a^2)$. To be consistent with the linearization, we must neglect such quadratic terms. This implies that we apply the surface b.c. at $z = 0$.

We seek plane wave solutions

$$\eta = a\, e^{-i\omega t+ikx} \qquad \text{and} \qquad \phi = A\, e^{-i\omega t+ikx} Z(z)$$

Notice that we cannot have a three-dimensional plane wave

$$\phi = A\, e^{i(kx+ly+mz-\omega t)}$$

Since the Laplacian operator would imply

$k^2+l^2+m^2 = 0$ which is impossible if all wave-number components are real.

Therefore the solutions are plane waves in two-dimensions in which ϕ has a vertical variation

$$\phi = A\, Z(z)\, e^{i(kx+ly-\omega t)}$$

We are for simplicity solving the one-dimensional case which can be immediately generalized to two dimensions.

The interior equation gives $-k^2 Z + Z_{zz} - 0$ that is $Z(z) = e^{\pm kz}$.

The linear combination of these two solutions that satisfies the boundary condition $\phi_z = 0$ at $z = -D$ is

$$Z(z) = \cosh[k(z+D)]$$

The two free surface b.c. can be combined

$$\frac{\partial}{\partial t}(\phi_t + g\eta) = 0 \;\rightarrow\; \phi_{tt} + g\eta_t = 0$$

But $\eta_t = \phi_z$ hence $\phi_{tt} + g\phi_z = o$ at $z = o$

The solution $\phi = A\, e^{i(kx-\omega t)} \cos[k(z+D)]$ satisfies the above equation at $z = 0$ if

$$\omega = \pm\sqrt{gk\tanh(kD)}$$

dispersion relation

$$\omega^2 = gK\tanh(kD)$$

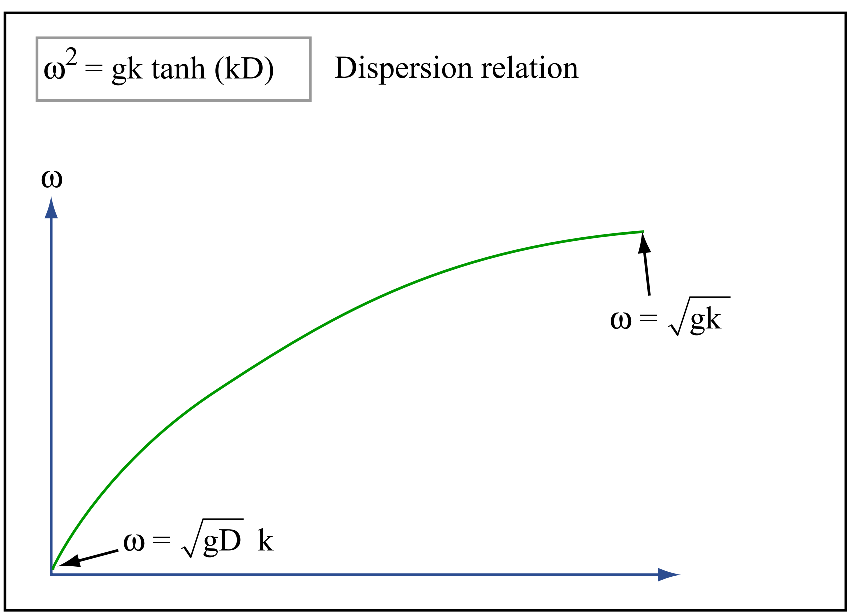

Figure by MIT OpenCourseWare.

Figure 2.

These waves are dispersive and can propagate in the ± x direction

From $\eta_t = \phi_z$ $\quad -i\omega a\, e^{i(kx-\omega t)} = A\, e^{i(kx-\omega t)}\, k\sinh(kD)$

from which $A = -\frac{i\omega a}{k\sinh(kD)}$ a, the amplitude of the free surface, is the reference term

A is complex. We must take this into account when writing the final solutions in real form:

$$\phi = \frac{-ia\omega}{k\sinh(kD)}\cosh[k(z+D)][\cos(kx-\omega t) + i\sin(kx-\omega t)]$$

Hence, taking the real part:

$$\eta = a\cos(kx-\omega t) \qquad (1)$$

$$\phi = \frac{a\omega}{k\sinh(kD)}\cosh[k(z+D)]\sin(kx-\omega t) \qquad (2)$$

$$u = \phi_x = \frac{a\omega}{\sinh(kD)}\cosh[k(z+D)]\cos(kx-\omega t) \qquad (3)$$

$$w = \phi_z = \frac{a\omega}{\sinh(kD)}\sinh[k(z+D)]\sin(kx-\omega t) \qquad (4)$$

Remember that: $\phi_t + \frac{p}{\rho} + gz + \frac{1}{2}|\nabla\phi^2| = 0$

linearizing: $\phi_t + \frac{p}{\rho} + gz = 0$

We applied this condition at z = η, that is z = 0 for the surface boundary condition but this is valid in general.

Hence:

$$p(z) = -\rho gz - \rho\phi_t$$

hydrostatic part

and

$$p(z) = -\rho gz \frac{+\rho\omega^2 a}{k\sinh(kD)}\cosh[k(z+D)]\cos(kx-\omega t) \qquad (5)$$

Notice that the pressure in a surface gravity wave is not hydrostatic but fluctuates around the hydrostatic background. As:

$$\omega^2 = gk\tanh(kD)$$

these waves are dispersive; the phase speed

$$c = \frac{\omega}{k} = \frac{\sqrt{gk\tanh(kD)}}{k} = \sqrt{\frac{g\tanh(kD)}{k}}$$

is different for different wavelengths. An initial "pattern made up of a superposition of plane waves will have each wave moving at a different phase speed and hence the pattern will DISPERSE. The phase speed is different from the group velocity

$$c_g = \frac{\partial\omega}{\partial k}$$

$$2\omega\frac{\partial\omega}{\partial k} = g[\tanh(kD) + \frac{kD}{\cosh^2(kD)}]$$

$$c_g = \frac{\partial\omega}{\partial k} = \frac{g[\tanh(kD) + \frac{kD}{\cosh^2(kD)}]}{2\sqrt{gk\tanh(kD)}}$$

Limiting cases

a) when the depth is very shallow or the wavelength is very long compared to the water depth we have shallow water waves:

$$D \ll \lambda \;\text{ or }\; kD \ll 1 \;\rightarrow\; \tanh(kD) \simeq kD$$

$$\omega^2 = (gD)k^2 \quad \text{or} \quad \omega = \pm\sqrt{gD}\;k$$

Then $c = \frac{\omega}{k} = \sqrt{gD}$; $c_g = \frac{\partial\omega}{\partial k} = \sqrt{gD}$; $c = c_g$

These waves are non-dispersive because c is the same for all of them and is equal to the group velocity.

b) when on the other side the wavelength is short compared with the depth, D>>λ: i.e. kD>>1

$$\tanh(kD) \to 1; \; \omega^2 = gK; \; \omega = \pm\sqrt{gk}$$

The deep water waves are dispersive

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \qquad c_g = \frac{\partial\omega}{\partial k} = \frac{g}{2\sqrt{gk}} = \frac{1}{2}\sqrt{\frac{g}{k}} \qquad c_g = \frac{1}{2}c$$

Consider a wave packet containing a short wave, Figs. 3.1 and 3.2. The amplitude will move with c_g while individual crests will move with c. As c_g = ½ c we will see individual crests appearing at the rear of the packet, moving through it to disappear at the leading edge of the packet. These waves are there even outside the packet, with a very small amplitude which first grows and then decays because they are modulated by the envelope of the packet. So it is the wave amplitude (energy) moving with c_g that has the physical content.

Let us look again at linearization and when it is valid considering the surface boundary condition.

Full b.c. $\phi_t + g\eta + \frac{1}{2}|\nabla\phi|^2 = 0$ at $z = \eta$

linearized $\phi_t + g\eta = 0$ at $z = o$

Now $\phi_t|_{z=\eta} = \phi_t|_{z=o} + \eta\phi_{tz}|_{z=o} +$

$$= [-i\omega A + a(-i\omega kA)e^{i(kx-\omega t)}]e^{i(kx-\omega t)}$$

Use expression in deep water for simplicity:

$$\phi = Ae^{kz}e^{i(kx-\omega t)}$$

$$\eta = ae^{i(kx-\omega t)}$$

Then $\eta\phi_{tz}|_{z=o} << \phi_t|_{z=o}$ provided that

$-i\omega kAa << -i\omega A$

or $ak<<1$ $\rightarrow$ Linearization is valid when the wave slope a/λ is small.

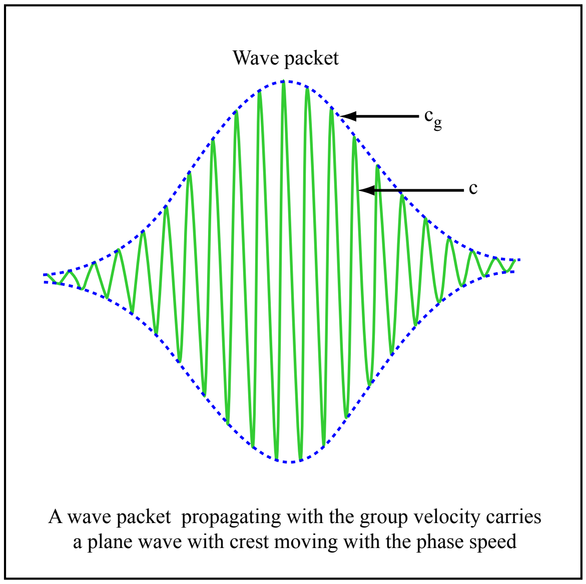

Figure by MIT OpenCourseWare.

Figure 3.

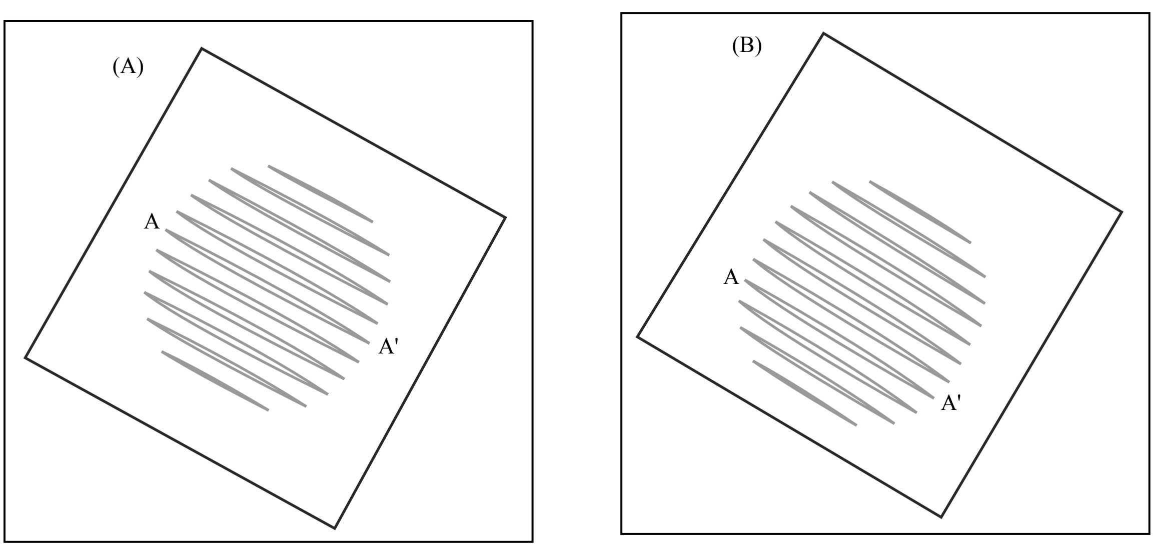

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Figure 4.

12.802 Wave Motion in the Ocean and the Atmosphere
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