

Laplace Tidal Equations — Vertical Structure

The vertical structure is given by the solution of

$$\bar{\alpha} \frac{\partial}{\partial z} \frac{1}{\bar{\alpha}(N^2 - \omega^2)} \frac{\partial}{\partial z} F + \left[\frac{\omega^2}{\bar{c}_s^2(N^2 - \omega^2)} - \frac{\partial}{\partial z} \frac{\omega^2}{g(N^2 - \omega^2)} \right] F = -\frac{1}{gH_{eq}} F$$

with

$$\left(\frac{\partial}{\partial z} - \frac{N^2}{g} \right) F = 0 \quad (\text{FixedB})$$

and/or

$$\left(\frac{\partial}{\partial z} - \frac{\omega^2}{g} \right) F = 0 \quad (\text{FreeB})$$

Low frequency

When ω is small, the VSE simplifies to

$$\bar{\alpha} \frac{\partial}{\partial z} \frac{1}{\bar{\alpha}N^2} \frac{\partial}{\partial z} F = -\frac{1}{gH_{eq}} F$$

with

$$\left(\frac{\partial}{\partial z} - \frac{N^2}{g} \right) F = 0 \quad (\text{FixedB})$$

and/or

$$\frac{\partial}{\partial z} F = 0 \quad (\text{FreeB})$$

In the case where $N^2H \ll g$, the boundary conditions become $F_z = 0$ at both types of surfaces. This equation clearly has a barotropic solution $F = 1$ with infinite equivalent depth. If we don't drop the N^2/g term, we can estimate H_{eq} by using $F \simeq 1 + f$ and integrating from a rigid bottom at $z = 0$ to a free surface at $z = H$. We find

$$H_{eq} = H \left[\frac{1}{H\bar{\rho}(0)} \int_0^H \bar{\rho}(z) dz \right] \simeq H$$

combining this with the horizontal equations gives us the dispersion relationship for long surface gravity waves

$$\omega^2 = f^2 + gH\mathbf{k}^2$$

and barotropic Rossby waves

$$\omega = -\frac{\beta k}{\mathbf{k}^2 + f^2/gH}$$

Generally, we deal with scales small compared to the external deformation radius \sqrt{gH}/f and can just use

$$\omega = -\frac{\beta k}{\mathbf{k}^2}$$

In addition, we have a set of internal modes. If we use a WKB approximation so that $F \simeq \exp(imz)$, we have

$$gH_{eq} \simeq N^2/m^2$$

and

$$\omega^2 = \frac{f^2 m^2 + N^2 \mathbf{k}^2}{m^2}$$

(long internal gravity waves) or

$$\omega = -\frac{\beta k}{\mathbf{k}^2 + m^2 f^2 / N^2}$$

(baroclinic Rossby waves).

Intermediate frequencies

When the frequency is comparable to N or f and $N^2 H \ll g$, $N^2 H^2 \ll \bar{c}_s^2$ we have

$$gH_{eq} \simeq (N^2 - \omega^2)/m^2$$

for the internal modes, leading to the full internal wave relationship

$$\omega^2 = \frac{f^2 m^2 + N^2 \mathbf{k}^2}{\mathbf{k}^2 + m^2}$$

High frequencies

When the frequencies are large, the vertical scales will be short and the VSE simplifies to

$$\frac{\partial^2}{\partial z^2} F + \frac{\omega^2}{\bar{c}_s^2} F = \frac{\omega^2}{gH_{eq}} F$$

with

$$\frac{\partial}{\partial z} F = 0 \quad (\textit{FixedB})$$

and/or

$$\left(\frac{\partial}{\partial z} - \frac{\omega^2}{g}\right) F = 0 \quad (\textit{FreeB})$$

For the internal modes,

$$gH_{eq} = \frac{\omega^2 \bar{c}_s^2}{\omega^2 - m^2 \bar{c}_s^2}$$

and the high-frequency horizontal equation gives

$$\omega^2 = gH_{eq} \mathbf{k}^2 = \frac{\omega^2 \bar{c}_s^2 \mathbf{k}^2}{\omega^2 - m^2 \bar{c}_s^2}$$

or

$$\omega^2 = \bar{c}_s^2(\mathbf{k}^2 + m^2)$$

—sound waves.

The external mode has $F \sim \exp(mz)$ with $m = \omega^2/g$ from the free surface boundary condition. The equivalent depth is

$$gH_{eq} = \frac{\omega^2 \bar{c}_s^2}{\omega^2 + m^2 \bar{c}_s^2}$$

Since these waves are still slow compared to the sound speed, this simplifies to

$$gH_{eq} = \frac{\omega^2 \bar{c}_s^2}{m^2 \bar{c}_s^2}$$

and the horizontal equation gives

$$\omega^2 = gH_{eq} \mathbf{k}^2 = \frac{\omega^2 \mathbf{k}^2}{m^2} = \frac{\omega^2 \mathbf{k}^2 g^2}{\omega^4}$$

or

$$\omega^2 = g|\mathbf{k}|$$

the dispersion relationship for short gravity waves.