

Non-acceleration result for stationary internal gravity waves

We have seen that for a 2-D flow (in x and z) the mean state is affected by waves through convergence of vertical fluxes of temperature ($\overline{\rho w' \theta'}$) and momentum ($\overline{\rho w' u'}$) by the waves. The overline denotes a zonal mean and the primes denote wave quantities. In this handout, we will derive expressions for how these fluxes vary in the vertical following Eliassen and Palm 1961.

We assume that the waves are stationary, inviscid, adiabatic and small amplitude. The assumption of small-amplitude waves allows us to use the linearized equations of motion to calculate the wave fluxes. The wave fluxes will change the mean state, but the changes in mean state are considered as a higher-order correction in our calculation of the fluxes.

Consider a basic state defined by zonal wind $U_0(z)$ and potential temperature $\theta_0(z)$. We assume the basic state is statically stable such that $\frac{\partial \theta_0}{\partial z} > 0$. Given the assumption of stationary waves ($\partial/\partial t = 0$), the linearized equations in log-pressure coordinates are:

$$U_0 \frac{\partial u'}{\partial x} + w' \frac{\partial U_0}{\partial z} + \frac{\partial \phi'}{\partial x} = 0, \quad (5)$$

$$\frac{\partial u'}{\partial x} + \frac{1}{\rho} \frac{\partial \rho w'}{\partial z} = 0, \quad (6)$$

$$U_0 \frac{\partial \theta'}{\partial x} + w' \frac{\partial \theta_0}{\partial z} = 0, \quad (7)$$

$$\frac{\partial \phi'}{\partial z} - \frac{R \Pi \theta'}{H} = 0. \quad (8)$$

Note that our equation numbering does not start from 1 to be consistent with the numbering used in class.

First consider the vertical wave flux of temperature ($\overline{\rho w' \theta'}$). Multiple Eq. 7 by θ' and take the zonal average (i.e., average in x):

$$\begin{aligned} U_0 \overline{\theta' \frac{\partial \theta'}{\partial x}} + \overline{w' \theta' \frac{\partial \theta_0}{\partial z}} &= 0, \\ \Rightarrow U_0 \frac{1}{2} \overline{\frac{\partial \theta'^2}{\partial x}} + \overline{w' \theta' \frac{\partial \theta_0}{\partial z}} &= 0, \\ \Rightarrow \overline{w' \theta' \frac{\partial \theta_0}{\partial z}} &= 0, \end{aligned}$$

where the last step follows because we assume the waves are periodic in x . Thus, we have zero vertical temperature flux ($\overline{\rho w' \theta'} = 0$) by our stationary and adiabatic waves.

Next we consider the vertical momentum flux ($\overline{\rho w' u'}$). This flux is generally not zero, but how does it vary in the vertical?

Multiply Eq. 5 by u' and average in x to give an equation that is the budget of kinetic energy of the waves (if we had not assumed stationary waves there would be a term $\partial(u'^2/2)/\partial t$):

$$\begin{aligned} U_0 \overline{\frac{\partial u'}{\partial x}} + \overline{w'u'} \frac{\partial U_0}{\partial z} + \overline{u' \frac{\partial \phi'}{\partial x}} &= 0 \\ \Rightarrow \overline{w'u'} \frac{\partial U_0}{\partial z} + \overline{u' \frac{\partial \phi'}{\partial x}} &= 0. \end{aligned}$$

Thus, the pressure force term (in the form of $\frac{\partial \phi'}{\partial x}$) in the zonal momentum equation causes $\overline{w'u'} \neq 0$ unlike for $\overline{w'\theta'}$. We next rewrite $\overline{u' \frac{\partial \phi'}{\partial x}}$ in terms of $\overline{\rho w' \phi'}$. Using property (iii) of zonal averages from class, we write that

$$\overline{u' \frac{\partial \phi'}{\partial x}} = -\overline{\phi' \frac{\partial u'}{\partial x}}.$$

Mass continuity (Eq. 6) then gives that

$$\begin{aligned} \overline{u' \frac{\partial \phi'}{\partial x}} &= \overline{\phi' \frac{1}{\rho} \frac{\partial \rho w'}{\partial z}} \\ &= \frac{1}{\rho} \overline{\frac{\partial \rho w' \phi'}{\partial z}} - \overline{\frac{\partial \phi'}{\partial z}} \overline{w'}. \end{aligned}$$

But hydrostatic balance (Eq. 8) implies that

$$\overline{\frac{\partial \phi'}{\partial z}} \overline{w'} = \frac{R\Pi}{H} \overline{\theta' w'} = 0,$$

and thus we have that

$$\overline{u' \frac{\partial \phi'}{\partial x}} = \frac{1}{\rho} \overline{\frac{\partial \rho w' \phi'}{\partial z}}.$$

The final form of our energy equation is

$$\boxed{\overline{\rho w' u'} \frac{\partial U_0}{\partial z} + \frac{\partial \overline{\rho w' \phi'}}{\partial z} = 0,} \quad (9)$$

where the first term represents conversion between wave and mean energy, and the second term is the divergence of the vertical wave energy flux. We have found a relation between the vertical momentum flux $\overline{\rho w' u'}$ and the vertical wave energy flux $\overline{\rho w' \phi'}$, but we will need another constraint to find either flux individually.

We go back to the wave zonal momentum equation (Eq. 5) and group the x -derivatives together:

$$\frac{\partial}{\partial x} (U_0 u' + \phi') + w' \frac{\partial U_0}{\partial z} = 0.$$

We next multiply by $U_0 u' + \phi'$ (last time we multiplied by u') to give

$$\frac{\partial}{\partial x} \left[\frac{1}{2} (U_0 u' + \phi')^2 \right] + U_0 w' u' \frac{\partial U_0}{\partial z} + w' \phi' \frac{\partial U_0}{\partial z} = 0.$$

Taking a zonal average then gives that

$$\frac{\partial U_0}{\partial z} (U_0 \overline{w' u'} + \overline{w' \phi'}) = 0.$$

In the case that $\frac{\partial U_0}{\partial z} \neq 0$, we multiple by ρ to give a second relation between the vertical wave momentum and energy fluxes:

$$\boxed{U_0 \rho \overline{w' u'} + \rho \overline{w' \phi'} = 0.} \quad (10)$$

If $\frac{\partial U_0}{\partial z} = 0$, we still get the same result as follows: Eq. 5 implies that

$$\begin{aligned} U_0 \frac{\partial u'}{\partial x} + \frac{\partial \phi'}{\partial x} &= 0 \\ \Rightarrow U_0 u' + \phi' &= 0, \end{aligned}$$

where the second equality follows because the waves have zero mean. Multiplying by $\rho w'$ and taking a zonal average gives Eq. 10 as required.

Substituting for $\rho \overline{w' \phi'}$ from Eq. 10 into Eq. 9 gives that:

$$\begin{aligned} \rho \overline{w' u'} \frac{\partial U_0}{\partial z} - \frac{\partial}{\partial z} (U_0 \rho \overline{w' u'}) &= 0 \\ \Rightarrow \rho \overline{w' u'} \frac{\partial U_0}{\partial z} - \frac{\partial U_0}{\partial z} \rho \overline{w' u'} - U_0 \frac{\partial}{\partial z} (\rho \overline{w' u'}) &= 0. \end{aligned}$$

Our final and simple result is that

$$\boxed{U_0 \frac{\partial}{\partial z} (\rho \overline{w' u'}) = 0.} \quad (11)$$

The vertical wave momentum flux $\rho \overline{w' u'}$ is constant in the vertical ($\partial \rho \overline{w' u'} / \partial z = 0$) except where $U_0 = 0$. Since we have previously shown that

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho} \frac{\partial}{\partial z} (\rho \overline{w' u'}),$$

we conclude that small-amplitude, adiabatic, inviscid, and stationary waves do not affect the mean flow except where $U_0 = 0$. We are considering stationary waves (phase speed zero), and thus levels with $U_0 = 0$ are “critical levels” defined by the phase speed being equal to U_0 .

We will consider the implications of this remarkable non-acceleration result in our next class.

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12.810 Dynamics of the Atmosphere

Spring 2023

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