Internal gravity waves[1]

In most places, and at most times, the atmosphere is stably stratified to unsaturated displacements. Here we consider what happens when a stably stratified fluid is perturbed. These introductory notes cover the simplest case of a Boussinesq fluid.

Boussinesq flow

We begin with the Boussinesq equations:

$$\begin{aligned}
\frac{Du}{Dt} &= -\frac{\partial \phi}{\partial x} \\
\frac{Dv}{Dt} &= -\frac{\partial \phi}{\partial y} \\
\frac{Dw}{Dt} &= -\frac{\partial \phi}{\partial z} + b \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{Db}{Dt} &= 0
\end{aligned} \tag{1}$$

where $\mathbf{u} = (u, v, w)$ is the velocity, b is the buoyancy, ϕ is the perturbation pressure divided by the reference density, and the Lagrangian derivative is $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$. Note that the third of (1) becomes the equation of hydrostatic balance when Dw/Dt is negligible. We will replace this equation by

$$\alpha \frac{Dw}{Dt} = -\frac{\partial \phi}{\partial z} + b \ .$$

The constant α is a trick: $\alpha = 1$, of course, but we shall carry it through the analysis so that we can, after the fact, look at the hydrostatic case by setting $\alpha = 0$.

Waves on a motionless basic state

Assume a motionless, stratified, basic state, with $u_0 = v_0 = w_0 = 0$, $b_0 = N^2 z +$ constant, $\phi_0 = \int b_0 \, dz$. $N^2 > 0$, so this state is stably stratified. Then we assume there are *small-amplitude* perturbations to the basic state denoted u', v', w', b' and ϕ' such that $b = b_0 + b'$ and similarly for the other variables. The perturbations

[1] These notes are adapted from notes courtesy of Alan Plumb

approximately satisfy the linearized equations resulting from the neglect of nonlinear terms such as $u'\frac{\partial u'}{\partial x}$ in (1):

$$
\begin{aligned}
\frac{\partial u'}{\partial t} &= -\frac{\partial \phi'}{\partial x} \\
\frac{\partial v'}{\partial t} &= -\frac{\partial \phi'}{\partial y} \\
\alpha\frac{\partial w'}{\partial t} &= -\frac{\partial \phi'}{\partial z} + b' \\
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} &= 0 \\
\frac{\partial b'}{\partial t} + N^2 w' &= 0
\end{aligned}
$$

Denoting the real part by Re, look for wavelike solutions of the form

$$
\begin{pmatrix} u' \\ v' \\ w' \\ \phi' \\ b' \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} U \\ V \\ W \\ \Phi \\ B \end{pmatrix} e^{i(kx+ly+mz-\omega t)}\right]
$$

Then

$$
\begin{aligned}
\omega U - k\Phi &= 0 \\
\omega V - l\Phi &= 0 \\
\omega\alpha W - m\Phi - iB &= 0 \\
kU + lV + mW &= 0 \\
-i\omega B + N^2 W &= 0
\end{aligned}
$$

From the last of these, $iB = N^2 W/\omega$, so the third eq. gives $(\omega\alpha - N^2/\omega)\, W - m\Phi = 0$. Substitute for U, V, W from the first three equations into the fourth equation to give

$$
\left[\frac{k^2}{\omega} + \frac{l^2}{\omega} + \frac{m^2}{(\omega\alpha - N^2/\omega)}\right]\Phi = 0 \ ,
$$

and hence

$$
\omega^2 = \frac{N^2\left(k^2+l^2\right)}{\left[\alpha\left(k^2+l^2\right)+m^2\right]} \ .
$$

Nonhydrostatic case $(\alpha = 1)$ For the general case, $\alpha = 1$, and the dispersion relation is

$$
\omega = \pm N\sqrt{\frac{k^2+l^2}{k^2+l^2+m^2}} \tag{2}
$$

Note that this can be written

$$\omega = \pm N \ \sin\gamma \ ,$$

where γ is the angle the wavenumber vector $\mathbf{k} = (k, l, m)$ makes with the vertical. So $|\omega| \leq N$.

The phase speed in the direction of $\mathbf{k}$ is given by

$$c = \frac{\omega}{|\mathbf{k}|}$$

and the group velocity is

$$\mathbf{c}_g = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right) = \frac{\omega m}{(k^2 + l^2 + m^2)}\left[\frac{km}{k^2 + l^2}, \frac{lm}{k^2 + l^2}, -1\right] .$$

Note:

1. $\mathbf{c}_g \cdot \mathbf{k} = 0$: group propagation is *along* the phase lines

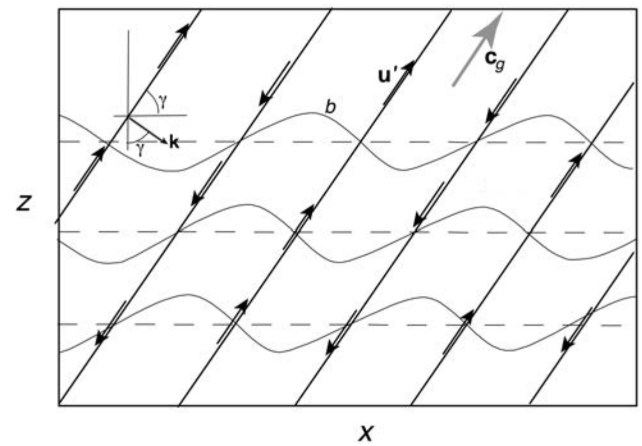
2. From the continuity eq., $\mathbf{k} \cdot \mathbf{u}' = 0$ — the fluid motions are along the phase lines. (Note that this implies no advection of wave properties; *e.g.*, since b' does not vary along lines of constant phase, $\mathbf{u}' \cdot \nabla b' = 0$. Hence the nonlinear advection terms we neglected on the grounds of small amplitude are in fact zero — a monochromatic plane internal gravity wave in a uniform medium is in fact a nonlinear solution to the problem!)

3. Note that point (2) implies that fluid motions are normal to $\mathbf{k}$. So as $\gamma \to \pi/2$, the motions are vertical and $\omega \to N$, the buoyancy frequency; as $\gamma \to 0$, the motions are horizontal (against which the stratification offers no resistance) and $\omega \to 0$.

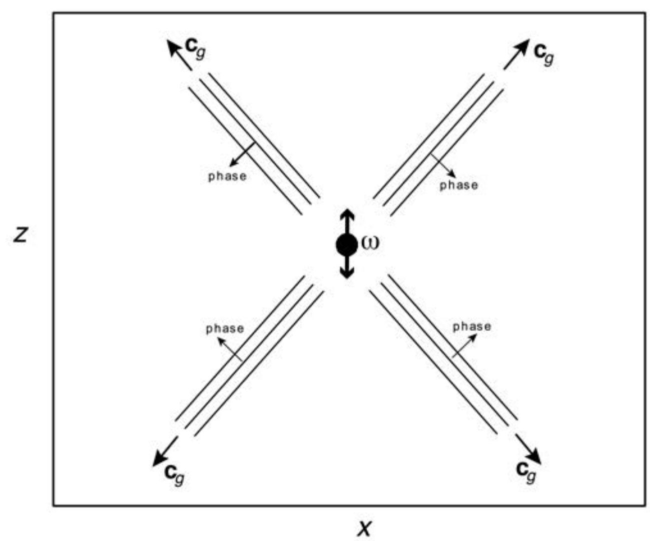
4. Note that if all components of $\mathbf{k}$ are real, $\omega \leq N$: disturbances with $\omega > N$ cannot propagate.

5. $(c_g)_x = m^2 k^2 c_x / \left[(k^2 + l^2 + m^2)(k^2 + l^2)\right]$, so the x components of phase and group velocities are in the same direction. Similarly, the y component. But $(c_g)_z = -m^2 c_z / (k^2 + l^2 + m^2)$ — the vertical components of group and phase velocities have *opposite* signs.

So an upward (and rightward) propagating wave looks as shown in the following figure:



From a localized source oscillating with a single frequency ω, the waves form rays (the "St Andrews' cross") at angles $\gamma = \sin^{-1}(\omega/N)$ to the horizontal, with the phase propagation *across* the rays:



Hydrostatic case $(\alpha = 0)$ When $\alpha = 0$, the dispersion relation becomes

$$\omega = \pm\frac{N}{m}\sqrt{k^2 + l^2} = \pm N \tan\gamma$$

There is no longer any restriction $\omega \leq N$, so the hydrostatic approximation is not valid for high frequency waves for which this approximation predicts $\omega \gtrsim N$, but it should be good for $\gamma \ll 1$ $(\omega \ll N)$. Equivalently, it requires $k^2 + l^2 \ll m^2$, *i.e.*, vertical scales much less than horizontal scales.

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