12.810 Problem set 1

Need help?: Office hours Thursdays 1.30-2.30pm in 54-1712.

Collaboration is allowed, but write up the solution on your own. Show all work. Give units for all numerical results. Put axis labels and units on any graphs.

1. In the small angle limit of the Held-Hou theory, the vertical-average potential temperature in radiative equilibrium is given by

$$\langle\theta_E\rangle(\phi) = \langle\theta_E\rangle(0) - \Delta_h\theta_0\phi^2 \tag{1}$$

and the vertical-average potential temperature in the angular-momentum conserving regime is

$$\langle\theta_m\rangle(\phi) = \langle\theta_m\rangle(0) - \frac{\Omega^2 a^2 \theta_0}{2gH}\phi^4. \tag{2}$$

Use the two matching conditions (continuity of temperature at the poleward boundary and energy conservation) to show that the poleward boundary of the Hadley cell is at the latitude

$$\phi_h = \left(\frac{5R}{3}\right)^{1/2}, \tag{3}$$

and the potential temperatures at the equator are related by

$$\langle\theta_m\rangle(0) = \langle\theta_E\rangle(0) - \frac{5}{18}R\Delta_h\theta_0, \tag{4}$$

where the thermal Rossby number is given by $R = gH\Delta_h/(\Omega^2 a^2)$.

2. In this problem you will derive a criterion for the onset of a zonally-symmetric circulation for a case in which the thermal forcing of the circulation is localized away from the equator. This is an idealized prototype problem for the onset of the monsoon.

Consider the radiative equilibrium temperature distribution $\theta_E(\phi)$ shown in Fig. 1 which has a "bump" localized off the equator. For a sufficiently weak thermal forcing (i.e. a small bump), it is possible to have a radiative-equilibrium (RE) solution with no circulation. However, a circulation must occur for a sufficiently strong thermal forcing. You will derive a sufficient condition on θ_E for the RE solution to have broken down.

(a) Assuming that the atmosphere is in radiative equilibrium, the nonlinear thermal wind balance is given by

$$\frac{\partial}{\partial z}\left[2\Omega u \sin\phi + \frac{\tan\phi}{a}u^2\right] = -\frac{g}{a\theta_0}\frac{\partial\theta_E}{\partial\phi}. \tag{5}$$

Integrate the nonlinear thermal wind balance in the vertical assuming weak winds near the surface, and show that the angular momentum m_H at the upper boundary satisfies

$$m_H^2 = \Omega^2\, a^4\, \cos^4\phi - \frac{a^2 gH\cos^3\phi}{\theta_0\sin\phi}\frac{\partial\langle\theta_E\rangle}{\partial\phi}, \tag{6}$$

where $\langle\theta_E\rangle$ is the vertically average of θ_E from $z = 0$ to $z = H$.

(b) For weak thermal forcing, m_H decreases in the poleward direction. Now imagine we increase the magnitude of the thermal forcing (the size of the bump in Fig. 1). For large enough thermal forcing, the gradient of m_H will be reversed and an extremum in m_H will be created that violates Hide's theorem (as well as making the flow inertially unstable.) Thus a circulation must occur. Note that we assume there is a stress-free upper boundary at $z = H$ so that the meridional gradient of m_H is key for Hide's theorem.

The critical thermal forcing occurs when $\partial m_H/\partial\phi = 0$. As discussed in class, $\partial m_H/\partial\phi = 0$ is exactly what is needed for a steady inviscid circulation that conserves angular momentum! Show that the condition for the RE solution to remain valid is:

$$-\frac{gH}{\theta_0}\frac{\partial}{\partial\phi}\left[\frac{\cos^3\phi}{\sin\phi}\frac{\partial\langle\theta_E\rangle}{\partial\phi}\right] < 4\Omega^2 a^2 \sin\phi\cos^3\phi. \quad (7)$$

(c) Assuming a narrow bump in θ_E, would the bump be more effective in driving a circulation if the bump is located at a low or a high latitude? Explain your reasoning.

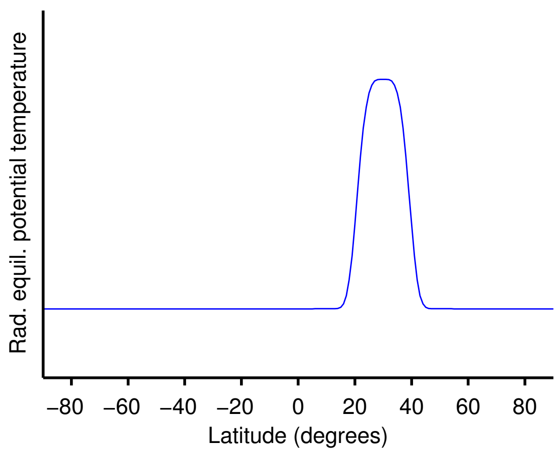

Figure 1: Radiative-equilibrium potential temperature $\langle\theta_E\rangle$ as a function of latitude. Gradients in $\langle\theta_E\rangle$ only occur away from the equator.

3. The Held-Hou theory uses angular momentum conservation and energy conservation to give a prediction for the poleward extent of the Hadley cell. However, the presence of eddies means that there is an export of energy from the Hadley cell to the extratropics and we can no longer assume that the Hadley cell is energetically closed. Here you will make an estimate for the poleward extent of the Hadley cell by assuming that it is at the latitude at which the circulation transitions from the Hadley-cell regime to the eddy-driven midlatitude regime. We estimate this as the latitude at which baroclinic instability begins to generate eddies (rather than using the matching conditions in the Held-Hou model).

(a) We will discuss baroclinic instability later in the course. For now, take it as given that the criterion for baroclinic instability is $u \geq c\beta g\Delta_v H/f^2$, where u is the zonal wind in the upper troposphere, c is an order-one constant, f is the Coriolis parameter, H is the tropopause height, β is the meridional gradient of the Coriolis parameter, and Δ_v is the vertical stratification parameter in radiative equilibrium in the Held-Hou theory.[1] Use this criterion to show that the latitude ϕ_h of the poleward edge of the Hadley cell is given by

$$\phi_h = \left(\frac{cg\Delta_v H}{2a^2\Omega^2} \right)^{\frac{1}{4}}. \tag{8}$$

You should assume that angular momentum conservation holds in the upper branch of the Hadley cell and make the small-angle approximation.

(b) Compare and contrast the dependencies of ϕ_h on the different parameters according to expression (8) versus the expression for ϕ_h from the Held-Hou theory. Why does ϕ_h from the Held-Hou theory not depend on Δ_v? Why does ϕ_h from (8) not depend on Δ_h?

(c) The Held-Hou theory assumes a zonally symmetric circulation. What kind of planet would not have eddies generated by baroclinic instability according to your result above?

[1] This criterion is derived from a two-layer model of baroclinic instability. We are also assuming that the vertical temperature stratification is set by radiation (or more realistically for the tropics by the combination of radiation and convection).