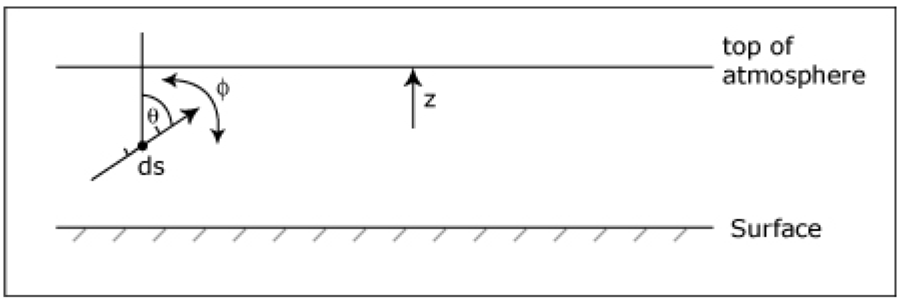
1. Equation of Radiative Transfer

Specific Intensity of Radiation Transfer

$$I_\nu(\theta,\phi,z) = \frac{E_\nu(\text{Energy})}{dA(\text{Area})\ d\Omega(\text{Solid Angle})\ d\nu(\text{frequency interval})\ dt(\text{time})}$$



In an interval ds, we lose intensity by extinction (scattering and absorption) and gain it by emission and scattering.

Lambert's Law: The extinction process is linear, independently in the intensity of radiation and in the amount of matter, provided that the physical state (i.e. temperature, pressure, composition) is held constant.

From Lambert's Law, the change of intensity along a path ds is proportional to the amount of matter in the path and to the intensity of radiation:

$$dI_\nu(\text{extinction losses}) = -\alpha_\nu I_\nu\, ds \quad \text{where } \alpha_\nu = \text{volume extinction coefficient} \qquad (1)$$

The argument that the extinction process is linear in the amount of matter applies with equal force to the emission process. Therefore, we write:

$$dI_\nu(\text{gains}) = \alpha_\nu J_\nu\, ds \qquad (2)$$

where we have defined the source function, J_ν.

The extinction coefficient can be expressed as the sum of an absorption coefficient (k_ν) and a scattering coefficient (σ_ν).

$$\alpha_\nu = k_\nu + \sigma_\nu \qquad (3)$$

The most general problem in atmospheric radiation, therefore, has a source function consisting of two parts,

$$\alpha_\nu J_\nu = k_\nu J_\nu \text{ (thermal)} + \sigma_\nu J_\nu \text{ (scattering)} \tag{4}$$

where $k_\nu J_\nu \text{ (thermal)} = \hat{\varepsilon}_\nu = \frac{\varepsilon_\nu}{4\pi}$ for isotropic emission.

$\sigma_\nu J_\nu \text{ (scat)}$, on the other hand, is given by two terms, describing the diffusely scattered radiation and the singly scattered incident beam of radiation (the sun).

$$\begin{aligned}\frac{dI_\nu}{ds}(\theta,\phi,z) &= -\alpha_\nu \left(I(\theta,\phi,z)\right) + \hat{\varepsilon}_\nu(\theta,\phi,z) \\ &+\sigma_\nu \int\int P(\theta,\phi,\theta',\phi') I(\theta',\phi',z)\frac{d\Omega}{4\pi} \\ &+\sigma_\nu \frac{\pi F}{4\pi}\exp\left(-\alpha_\nu z/\cos\theta_0\right) P(\theta,\phi,\theta_0,\phi_0)\end{aligned} \tag{5}$$

where $P(\theta,\phi,\theta',\phi')$ is the scattering phase function (or scattering diagram) and is normalized such that $\int_{4\pi} P\frac{d\Omega}{4\pi} = 1$ where $d\Omega$ is an element of solid angle. The shape of the phase function can be usefully characterized by a single number, $<\cos\eta> = \int_{4\pi}(\cos\eta)p\frac{d\Omega}{4\pi}$ where η is the scattering angle and $<\cos\eta>$ is called the asymmetry parameter (which varies between 1 and -1 and is 0 for isotopic scattering.

Dividing by $\alpha_\nu(z)$, we have:

$$\begin{aligned}\frac{1}{\alpha_\nu(z)}\frac{dI_\nu}{ds}(\theta,\phi,z) &= -I_\nu(\theta,\phi,z) + \frac{\hat{\varepsilon}_\nu(z)}{\alpha_\nu(z)} \\ &+\frac{\sigma_\nu(z)}{4\pi\ \alpha_\nu(z)}\int\int P(\theta,\phi,\theta',\phi') I(\theta',\phi',z)\ \sin\theta' d\theta' d\phi' \\ &+\frac{\sigma_\nu(z)\pi F}{\alpha_\nu(z)4\pi}\exp\left(-\alpha_\nu(z)\ z/\cos\theta_0\right) P(\theta,\phi,\theta_0,\phi_0)\end{aligned} \tag{6}$$

Let us now introduce the following definitions:

$$d\tau_\nu = -\alpha_\nu dz \qquad ds = \frac{dz}{\mu} \qquad \mu = \cos\theta \tag{7}$$

where τ_ν = <u>vertical optical depth</u> measured from the top of the atmosphere $(\tau_\nu = 0 \text{ at } z = \infty)$. *Note that this coordinate differs for each* ν.

We then obtain the following Radiative Transfer Equation in differential form in a "plane parallel" atmosphere:

$$\mu \frac{dI(\tau_\nu,\mu,\phi)}{d\tau_\nu} = I(\tau_\nu,\mu,\phi) - \frac{\hat{\varepsilon}_\nu(\tau_\nu)}{\alpha_\nu(\tau_\nu)} - \frac{\sigma_\nu(\tau_\nu)}{4\pi\alpha_\nu(\tau_\nu)} \int\int P(\theta,\phi,\theta',\phi') I(\tau_\nu,\theta',\phi') \sin\theta' \, d\theta' \, d\Phi$$

$$- \frac{\sigma_\nu(\tau_\nu)\pi F}{4\pi\alpha_\nu(\tau_\nu)} \exp\left(-\alpha_\nu(\tau_\nu) z / \cos\theta_0\right) P(\theta,\phi,\theta_0,\phi_0) \qquad (8)$$

or

$$\mu \frac{dI(\tau_\nu,\mu,\phi)}{d\tau_\nu} = I(\tau_\nu,\mu,\phi) - J(\tau_\nu,\mu,\phi)$$

where we have defined the source function, $J(\tau_\nu,\mu,\phi)$ as:

$$J(\tau_\nu,\mu,\phi) = \frac{\hat{\varepsilon}_\nu}{\alpha_\nu} + \frac{\tilde{\omega}_\nu}{4\pi} \int\int P(\mu,\phi,\mu',\phi') I(\tau_\nu,\mu,\phi,\mu',\phi') \, d\mu' d\phi'$$

$$+ \frac{\tilde{\omega}_\nu \, \pi F_\nu}{4\pi} \exp(\tau_\nu/\mu_0) P(\mu,\phi,\mu_0,\phi_0) \qquad (9)$$

and $\frac{\sigma_\nu}{\alpha_\nu} = \tilde{\omega}_\nu$ = **Single Scattering Albedo.**

Let us also be reminded that the atmosphere in general contains both gases and particulates (aerosols). Each has scattering and absorption properties that we need to consider. Thus we have:

$$\alpha_\nu = k_\nu(\text{gases}) + k_\nu(\text{aerosols}) + \sigma_\nu(\text{gases}) + \sigma_\nu(\text{aerosols})$$

There are many cases where the physical situation enables the neglect of one or more of these terms and a related simplification of the Radiative Transfer Equation. We will now examine a few such cases.

Let us first apply Kirchoff's Law to our Radiative Transfer Equation. "The ratio of emission and fractional absorption in any direction of a slab of any thickness in thermodynamic equilibrium equals the black body intensity." So – in a non-scattering atmosphere, we have

$$\frac{\hat{\varepsilon}_\nu(\theta,\phi,\tau_\nu)}{\alpha_\nu} = \frac{\varepsilon_\nu(\tau_\nu)/4\pi}{\alpha_\nu} = B_\nu(\tau_\nu) = \text{Black Body Function}$$

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\left(e^{h\nu/kT-1}\right)} \qquad B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{\left(e^{hc/\lambda kT-1}\right)}$$

where $B_\nu(\tau_\nu)$ is isotropic

This gives us:

$$J_\nu(\tau_\nu,\mu,\phi) = B_\nu(\tau_\nu) + \frac{\tilde{\omega}_\nu}{4\pi}\int\int P(\mu,\phi,\mu',\phi') I(\mu,\phi,\mu',\phi',\tau_\nu)\, d\mu\, d\phi$$

$$+\frac{\tilde{\omega}_\nu \pi F}{4\pi}\exp\left(\tau_\nu/\mu_0\right) P(\mu,\phi,\mu_0,\phi_0) \qquad (9a)$$

Also if all relevant properties ($\alpha_\nu, \tilde{\omega}_\nu, B_\nu(\tau_\nu)$) are horizontally invariant, then I is independent of azimuth angle, ϕ – i.e. $I_\nu(\tau_\nu,\mu,\phi) = I_\nu(\tau_\nu,\mu)$

and we have:

$$\mu\frac{dI_\nu(\mu,\tau_\nu)}{d\tau_\nu} = I_\nu(\mu,\tau_\nu) - J(\tau_\nu) \qquad (10)$$

Case I:

Let us first consider the simplest case characterized as follows:

a) We observe sunlight of visible wavelengths through a non-scattering atmosphere:

$B_\nu(\tau_\nu) << \frac{\pi F}{4\pi}$ and also $B_\nu(\tau_\nu) << I_\nu(\mu,\tau_\nu)$

or

b) Bright, artificial source of radiation shining through an atmospheric path (e.g.- laser):

We then have that $I_\nu(\mu,\tau_\nu) >> J_\nu(\tau_\nu)$ and the Radiative Transfer Equation is simplified to:

$$\mu\frac{dI_\nu(\mu,\tau_\nu)}{d\tau_\nu} = I_\nu(\mu,\tau_\nu) \qquad (11)$$

and the solution of this equation is:

$\int_{I_o}^{I}\frac{dI_\nu}{I_\nu} = \int_0^{\tau_\nu}\frac{d\tau_\nu}{\mu_0}$ or $I_\nu(\mu_0,\tau_\nu) = I_{0\nu}(\mu_0,0)\exp\left(\tau_\nu/\mu_0\right)$

and since $\mu_0 = -\cos\theta_0$, we have

$I_\nu(\mu_0,\tau_\nu) = -I_{0\nu}(\mu_0,0)\exp - \left[\tau_\nu/\cos\theta_0\right]$

Beer's Law

Bouguer's Law
Lambert's Law

If we have $I(\theta_o, 0)$ as a slowly varying function of frequency, ν, and

We define the transmission function, $t = \exp - \left[\tau_\nu / \cos\theta_0 \right]$

$$\int_{\Delta\nu} I(\mu, \tau_\nu)\, d\nu = I(\theta_0, 0) \int_{\Delta\nu} \exp - \left[\tau_\nu / \cos\theta_0 \right] d\nu$$

$$= I(\theta_o, 0)\, \bar{t}_{\Delta\nu}$$

A lot of work has been done to develop methods for computing this mean transmission and we will return to this topic and examine it in detail a little later.

Case II:

Let us consider a cloudless atmosphere and the infrared portion of the electromagnetic spectrum. Due to the approximate separation of the solar emission spectrum and planetary emission spectrum (as discussed previously by Prof. Prinn), we now have:

$$J(\tau_\nu) \cong B(\nu, \tau_\nu)$$

And the Radiative Transfer Equation reduces to:

$$\mu \frac{dI(\mu, \tau_\nu)}{d\tau_\nu} = I(\mu, \tau_\nu) - B(\mu, \tau_\nu) \qquad (12)$$

This is a linear first order equation. If we apply $e^{-\tau/\mu}$ as an integrating factor, we obtain the following equation:

$$e^{-\tau/\mu} \frac{dI}{d\tau} - I \frac{e^{-\tau/\mu}}{\mu} = -\frac{B}{\mu} e^{-\tau/\mu} = \frac{d\left[I e^{-\tau/\mu} \right]}{d\tau} \qquad (13)$$

Lets consider the upward intensity at a level, z ($\mu > 0$). The origin of optical depth is at the top of the atmosphere and we will need to integrate from the level, z, to the surface. It is therefore convenient to change the variable of integration to $\tau' = \tau - \tau_z$ as we must integrate over optical depths ranging from zero to the optical depth at the surface of the earth. Thus, we have Eq. 14:

$$I e^{-\tau'/\mu} \Big|_{\tau_z}^{\tau_s} = \int_{\tau_z}^{\tau_s} \frac{-B(\tau')}{\mu} e^{-\tau'/\mu} d\tau' = \int_{\tau_z}^{\tau_s} \frac{-B(\tau - \tau_z)}{\mu} e^{-(\tau - \tau_z)/\mu} \, d\tau \qquad (14)$$

and finally the solution we are seeking:

$$I(\tau_z) = I(\tau_s) e^{-(\tau_s - \tau_z)/\mu} + \int_{\tau_z}^{\tau_s} B(\tau) e^{-(\tau - \tau_z)/\mu} \frac{d\tau}{\mu} \qquad (15)$$

where the subscript, s, in Eqs. 14 and 15 refers to the surface of the earth.

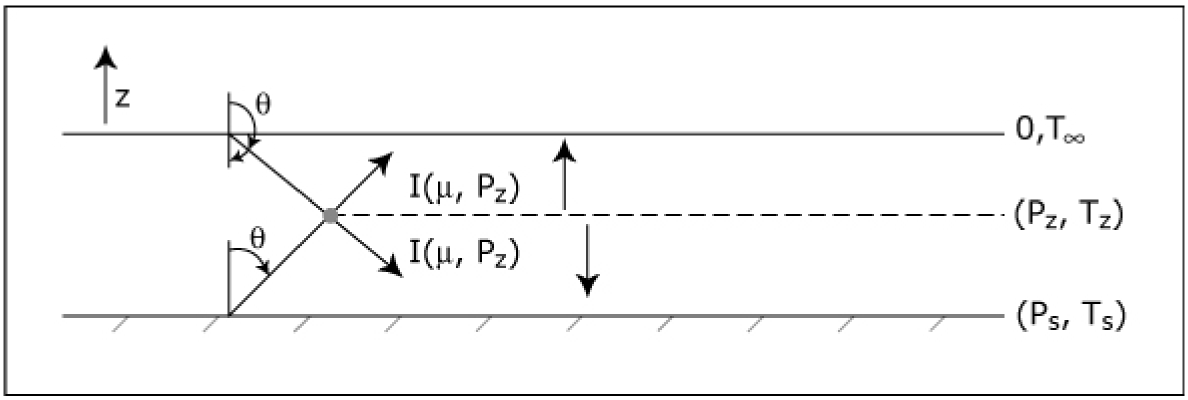
And, at the ground we typically have nearly black body emission in the infrared, so $I(\tau_s)$ can be replaced by $\varepsilon B(\tau_s)$ where ε is an emissivity (near unity) and $B(\tau_s)$ is the Planck Black Body Function.

And the downward solution is similarly given by:

$$I(\tau_z)\downarrow = -\int_0^{\tau_z} B(\tau)\, e^{-(\tau_z-\tau)/\mu}\, \frac{d\tau}{\mu} \tag{16}$$

In these equations we've dropped the frequency specification for simplification. But – we'll need to keep in mind that optical depth, Planck function and radiation intensity always depend on frequency (or wavelength).

If the temperature is known throughout the atmosphere, an exact solution is possible: i.e. – $B(\tau) = B(T(p))$



Consider pressure coordinates for which

$$d\tau = -\alpha\ dz = -k\,\rho_a\, dz = -k\,\chi_a\,\rho\ dz$$

$$= \frac{k\,\chi_a\ dp}{g} \quad \left(\text{since } \frac{dp}{dz} = -\rho g\right)$$

where k_ν is the absorption cross-section (in cm^2/gm), ρ_a is the absorber mass density and χ_a is mass mixing ratio of absorber, a.

Now, lets go back to the formal solution of the R.T.Eq. (Eq. 15) and examine the details:

$$\left.\begin{aligned} &B(\tau) \Rightarrow B(T_p) \\ &e^{-(\tau_s-\tau_z)/\mu} \Rightarrow \exp - \int_{p_z}^{p_s} \frac{k\,\chi_a}{g\,\mu} dP \\ &e^{-(\tau_z-\tau)/\mu} \Rightarrow \exp - \int_{p_z}^{p} \frac{k\,\chi_a}{g\,\mu} dP \end{aligned}\right\} \equiv \text{Transmission}$$

and if the ground can be considered black (a reasonable assumption in much of the thermal infrared), we have $\varepsilon B(\nu, T) \Rightarrow B(\nu, T_g)$

So – finally, we have:

$$I(\mu, P_z)\uparrow = B(T_g)\ \exp - \int_{p_z}^{p_s} \frac{k\,\chi_a}{g\,\mu} dp \ -\ \text{ground contribution}$$

$$+\int_{p_z}^{p_s} \frac{k\,\chi_a}{g\,\mu} B(T_p)\ \exp - \left[\int_{p_z}^{p} \frac{k\,\chi_a}{g\,\mu} dp'\right] dp \ -\ \text{atmospheric contribution}$$

(17)

For downward radiation $I(\mu, \tau)\downarrow$, we similarly have:

$$I(\mu, p_z)\downarrow = \int_{0}^{p_z} \frac{k\,\chi_a}{g|\mu|}\ B(T_p) \exp - \left[\int_{p}^{p_z} \frac{k\,\chi_a}{g|\mu|}\ dp'\right] dp \tag{18}$$

Let us recall that we defined the transmission function as follows:

$$t = \exp - \left[\int_{p_z}^{p} \frac{k\,\chi_a}{g\,\mu}\ dp\right]$$

From above, we have

$$I(\mu, p_z)\downarrow = -\int_{1}^{t_\infty} B(T_p)\, dt \tag{19}$$

where T = T(t)

with a similar expression for the upward intensity:

$$I(\mu, p_z)\uparrow = -\int_{1}^{t_s} B(T)\, dt + B(T_s)\, t_s \tag{20}$$

where t = transmission

In practice molecular absorption by atmospheric gases (H_2O, CO_2, O_3, N_2O, CO, CH_4, O_2, etc.) fluctuate rapidly with respect to frequency compared with the Planck function $B(\nu, T)$.
Therefore, when we consider a spectral interval appropriate for measurement, we can take the frequency variation into account and we have:

$$I_{\Delta\nu}\downarrow = \int_{\Delta\nu} I_\nu\,(\mu, p_z)\, d\nu = \int_{1}^{t_\infty} B(\bar{\nu}, \bar{T})\, d\bar{t}$$

$$\text{where } \bar{t} = \int_{\Delta\nu} \exp - \left\{\int_{p}^{p_z} \frac{k_\nu\,\chi_a\, dp'}{g|\mu|}\right\} d\nu \tag{21}$$

and this frequency averaged transmission represents the object of many years of work in atmospheric radiative transfer by many people. We will discuss this more completely when we discuss the HITRAN & MODTRAN transmission and radiation models later.

Case III:

Let us go back to Eq. 9a and consider the source function under conditions when the solar and diffuse radiation field is much greater than the Planck emission. The approximate separation of solar radiation and planetary emission will again be invoked to examine the radiative transfer problem in the visible portion of the spectrum where $F_\nu(\theta) >> B(\nu, \tau_\nu)$

and the scattered radiation,

$$I(\theta', \phi, \tau_\nu) = >> B(\nu, \tau_\nu)$$

As before, the formal solution is:

$$I(\tau, \mu, \mu_0, \phi, \phi_0)\uparrow = \int_0^{\tau} J(\tau', \mu, \mu_0, \phi, \phi_0)\, e^{-(\tau-\tau')/\mu}\, d\tau'/\mu \qquad \mu > 0$$

$$I(\tau, \mu, \mu_0, \phi, \phi_0)\downarrow = -\int_{\tau}^{\tau_0} J(\tau', \mu, \mu_0, \phi, \phi_0)\, e^{-(\tau'-\tau)/\mu}\, d\tau'/\mu \qquad \mu < 0$$

(22)

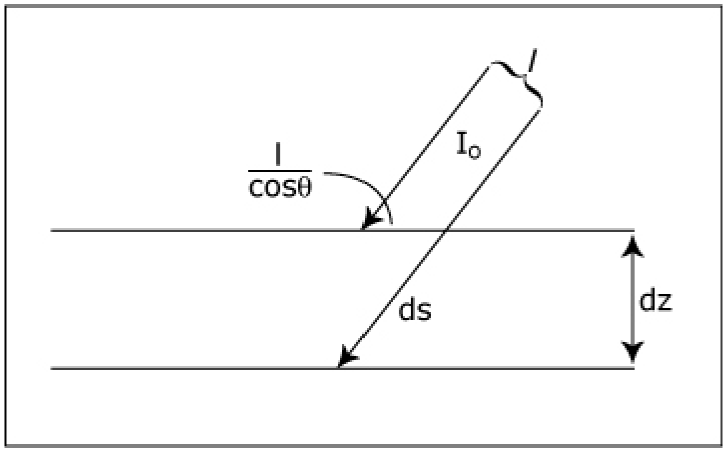
$$\text{where } J(\tau', \mu, \mu_0, \phi, \phi_0) = \frac{\tilde{\omega}_\nu}{4\pi}\int\int P(\mu, \phi, \mu', \phi')\, I(\tau', \mu, \phi, \mu', \phi')\, d\mu'\, d\phi' + \bar{\omega}_\nu \frac{F_\nu}{4\pi} \exp\left(-\tau_\nu/\mu_0\right) P(\mu, \phi, \mu_0, \phi_0)$$

The solution to this problem requires knowledge of the distribution of scatterers, the optical properties of the scatterers and their Phase Function (the probability that a photon incident from a particular direction will be scattered into another specific direction). In general, we must also deal with the complex problem of multiple scattering. We'll investigate the process of Mie Scattering and Absorption by spherical particles, having specified sizes and optical properties. We'll use a computer program that provides exact solutions for Mie Scattering and Absorption. Then, we'll be dealing with a computer model capable of computing the radiation field for multiple scattering, using a procedure known as Discrete Ordinates which divides the radiation field into Fourier components and integrates the set of independent equations using Gaussian quadrature.

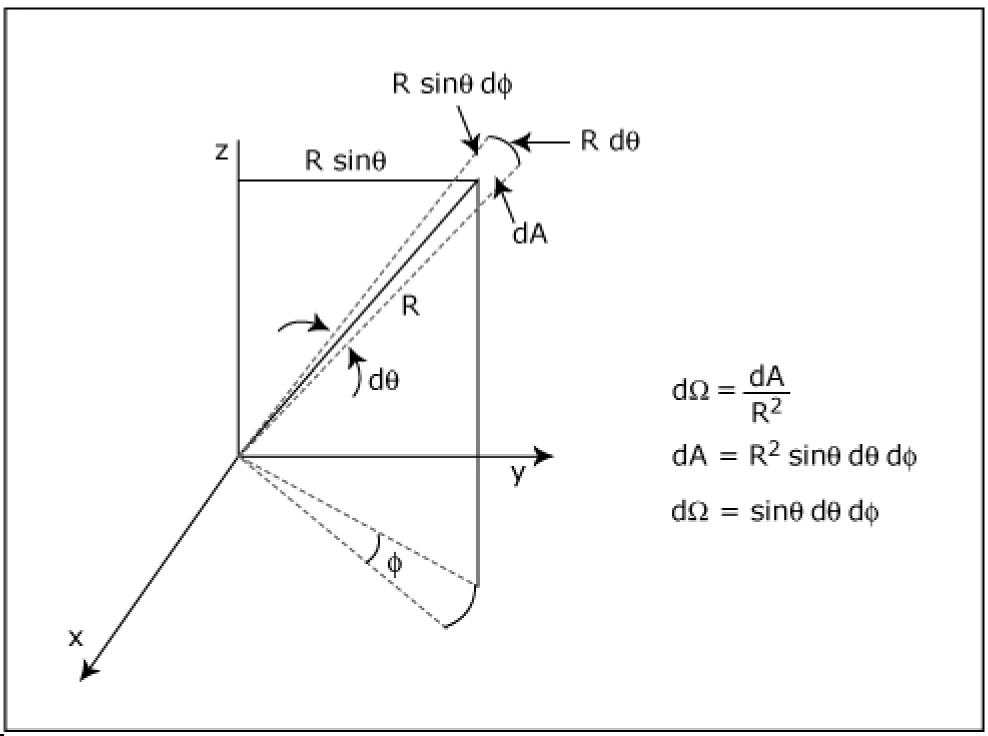
2. Radiation Intensity and Radiation Flux

a. Radiation Intensity – amount of energy per unit time contained in an element of solid angle which flows through a cross section of unit area perpendicular to the direction of the beam.

   Let us consider that we have isotropic radiation of intensity I_o falling on one face of a horizontal slab:



Solid Angle (in spherical coordinates)



b. Radiation Flux – amount of energy per unit time crossing a unit surface perpendicular to the z direction.

$$F \downarrow = \int_0^{2\pi} \int_0^{\pi/2} I \sin\theta \, d\theta \, d\phi = \int_0^{2\pi} \int_0^{\pi/2} I_o \sin\theta \, \cos\theta \, d\theta \, d\phi$$

$$= 2\pi I_0 \int_0^{\pi/2} \cos\theta \, \sin\theta \, d\theta = \pi I_0$$

For Isotropic Radiation, the flux is π times the intensity of a straight beam.

Applications using Radiation Intensity
1. Remote sounding
2. Satellite measurements
3. Target detection over horizontal/vertical paths

Applications using Flux:
1. Heating/Cooling of atmosphere
2. Radiation effects on climate.

3. Approximate Solution for Planetary Radiation

From Eq. 12, we have:

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - B_\nu \tag{23}$$

This can be transformed into an integral equation by integrating both sides over all angles:

$$\int_0^{2\pi}\int_{-1}^{1} \frac{dI_\nu}{d\tau_\nu} \mu \, d\mu \, d\phi = \int_0^{2\pi}\int_{-1}^{1} I_\nu \, d\mu \, d\phi - 4\pi B_\nu$$

$$\underbrace{\frac{d}{d\tau_\nu}\left(2\pi \int_{-1}^{1} \mu I_\nu \, d\mu\right)}_{\text{divergence of net upward flux}} = \underbrace{2\pi \int_{-1}^{1} I_\nu \, d\mu}_{\text{total flux } (4\pi\langle I_\nu \rangle)} - \underbrace{4\pi B_\nu}_{\text{total flux in an LTE enclosure}}$$

$$\left[\frac{d}{d\tau_\nu}(\pi F_\nu)\right]$$

$$\left[\begin{array}{l}\text{If } I_\nu = I_0 = \text{constant} \\ \int_{-1}^{1} I_\nu d\mu = 2\bar{I}_\nu\end{array}\right]$$

leading to the radiative transfer equation in net-flux form:

$$\frac{1}{4}\frac{dF_\nu}{d\tau_\nu} = \langle I_\nu \rangle - B_\nu \tag{24}$$

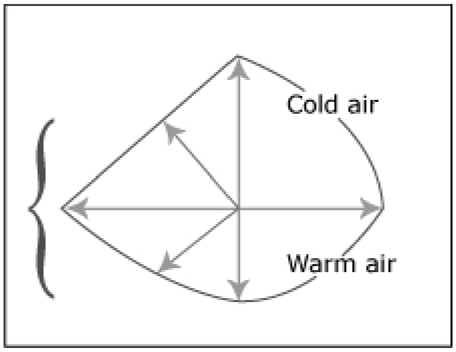
Now – multiply both sides by μ and integrate overe all angles:

$$\int_0^{2\pi}\int_0^{1} \mu^2 \, dI_\nu / d\tau_\nu \, d\mu \, d\phi = \int_0^{2\pi}\int_{-1}^{+1} \mu I_\nu \, d\mu \, d\phi - B_\nu \underbrace{\int_0^{2\pi}\int_{-1}^{1} \mu \, d\mu \, d\phi}_{0}$$

$$\underbrace{\frac{d}{d\tau_\nu}\left[2\pi \int_{-1}^{1} \mu^2 I_\nu \, d\mu\right]}_{\pi K_\nu} = \underbrace{\pi F_\nu}_{\text{net upward flux}}$$

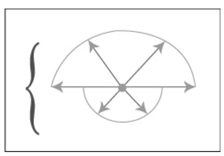
$$\boxed{\frac{dK_\nu}{d\tau_\nu} = F_\nu} \tag{25}$$

Now, I_ν looks something like this



In order to solve these equations, <u>Eddington</u> proposed a <u>two-stream</u> approximation:

$$I_\nu(\mu) \begin{cases} = I_\nu^+ & 0 \le \mu \le 1 \\ = I_\nu^- & 0 > \mu \ge 1 \end{cases}$$



Thus

$$4\pi\langle I_\nu \rangle = \int_0^{2\pi}\int_{-1}^{1} I_\nu \, d\mu \, d\phi \approx 2\pi\left(I_\nu^+ + I_\nu^-\right) \tag{26a}$$

$$\pi F_\nu = \int_0^{2\pi}\int_{-1}^{1} \mu \, I_\nu \, d\mu \, d\phi \approx \pi\left(I_\nu^+ + I_\nu^-\right) \tag{26b}$$

$$\pi K_\nu = \int_0^{2\pi}\int_{-1}^{1} \mu^2 \, I_\nu \, d\mu \, d\phi \approx \tfrac{2}{3}\pi\left(I_\nu^+ + I_\nu^-\right) \tag{26c}$$

$$\frac{1}{4}\frac{dF_\nu}{d\tau_\nu} = \tfrac{1}{2}\left(I_\nu^+ + I_\nu^-\right) - B_\nu \tag{26d}$$

where we have used Eq. 24.

$$\frac{2}{3}\frac{d}{d\tau_\nu}\left(I_\nu^+ + I_\nu^-\right) = F_\nu \tag{27}$$

where we have used Eq. 25.

Or, differentiate Eq. 26d and use Eq. 27:

$$\frac{1}{4}\frac{d^2F_\nu}{d\tau_\nu^2} = \frac{1}{2}\frac{d}{d\tau_\nu}\left(I_\nu^+ + I_\nu^-\right) - \frac{dB_\nu}{d\tau_\nu}$$

$$\frac{1}{4}\frac{d^2F_\nu}{d\tau_\nu^2} = \frac{3}{4}F_\nu - \frac{dB_\nu}{d\tau_\nu}$$

$$\boxed{\frac{d^2F_\nu}{d\tau_\nu^2} - 3F_\nu = -4\frac{dB_\nu}{d\tau_\nu}} \qquad (28)$$

This is known as Eddington's Equation.

The two required boundary conditions are usually given in the form of I^- or I^+. We have from Eq. 26d.

$$\left.\begin{aligned}\frac{1}{2}I_\nu^+ &= \frac{1}{4}\frac{dF_\nu}{d\tau_\nu} + B_\nu - \frac{1}{2}I_\nu^- \\ &= \frac{1}{4}\frac{dF_\nu}{d\tau_\nu} + B_\nu - \frac{1}{2}\left(I_\nu^+ - F_\nu\right)\end{aligned}\right\} \text{Using 26d}$$

where we have made use of Eq. 26b.

Thus: $I_\nu^+ = \frac{1}{4}\frac{dF_\nu}{d\tau_\nu} + B_\nu + \frac{1}{2}F_\nu$

$= E_\nu^s B_\nu^s$ (at bottom eg cloud-top or surface)

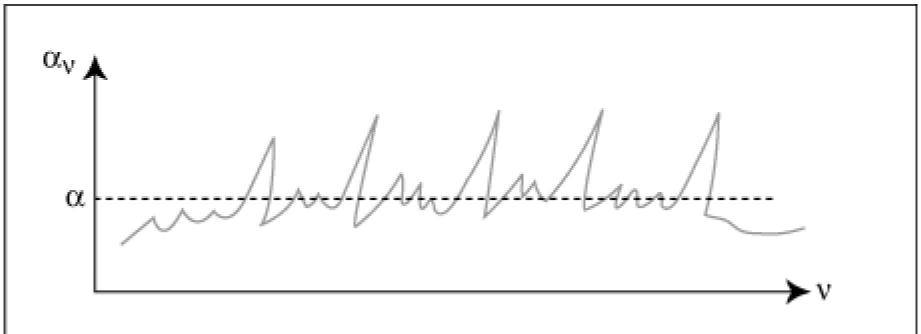
$= F_\nu$ (at top since $I_\nu^- = 0$ at $\tau_\nu = 0$)

and $I_\nu^- = I_\nu^+ - F_\nu$

$= \frac{1}{4}\frac{dF_\nu}{d\tau_\nu} + B_\nu - \frac{1}{2}F_\nu$ (From I^+ equation above)

= 0 (at top since no downward diffuse radiation)

To provide some simple analytical solutions it is useful to consider the **grey approximation** wherein α_ν is replaced by α (= grey absorption coefficient).



5 µm - 100 µm region where planetary radiation is high.

Thus, equation of radiative transfer can be integrated over frequency since

$$d\tau_\nu = -\alpha_\nu\, dz \Rightarrow d\tau = -\alpha\, dz$$

which is now independent of ν. Using the notation $\overline{(\ )} = \int (\ \ )\, d\nu$

$$\mu \overline{\frac{dI}{d\tau}} = \bar{I} - \bar{J}$$

$$\frac{d^2\bar{F}}{d\tau^2} - 3\bar{F} = -4\overline{\frac{dB}{d\tau}}$$

$$\bar{I}^+ = \frac{1}{4}\frac{d\bar{F}}{d\tau} + \bar{B} + \frac{1}{2}\bar{F} \quad \left(= E_s B_s \text{ at bottom or } \bar{F} \text{ at top}\right)$$

$$\bar{I}^- = \frac{1}{4}\frac{d\bar{F}}{d\tau} + \bar{B} - \frac{1}{2}\bar{F} \quad \left(= 0 \text{ at top}\right).$$ Also note that $B = \frac{ST^4}{\pi}$ from earlier lecture.

where s=Stefan's constant.

Example: Suppose we have an atmosphere at rest (i.e. no dynamical or latent heat fluxes). Suppose also that net radiative heating is zero everywhere – that is the net upward flux $\pi\bar{F}$ = constant (i.e. non-divergent). This state is called radiative equilibrium. Eddington's equation is now:

$$F = \frac{4}{3}\frac{d\bar{B}}{d\tau}$$

$$\bar{I}^+ = B + \frac{1}{2}\bar{F} \left(= \bar{F} \text{ at top}\right)$$

$$\bar{I}^- = B - \frac{1}{2}\bar{F} \left(= 0 \text{ at top}\right) \text{ or } \bar{B} = \frac{1}{2}\bar{F} \text{ at top}$$

i.e. $\bar{B}(0) = \frac{1}{2}\bar{F}$

$$\int_{\beta(0)}^{\bar{\beta}(\tau)} d\bar{B} = \frac{3}{4}$$

$$\bar{B}(\tau) - \bar{B}(0) =$$

$$\bar{B}(\tau) - \frac{1}{2}\bar{F} = \frac{3}{4}\,\bar{F}\,\tau$$

$$\bar{B}(\tau) = \bar{F}\left(3/4\,\tau + \frac{1}{2}\right)$$

At the top of the atmosphere we need for the planetary average:

<u>net incoming solar flux</u> = <u>net outgoing planetary flux</u>

$$\frac{(1-A)\,S\pi a^2}{4\pi a^2} = \pi F$$

$$\bar{F} = \frac{(1-A)\,S}{4\pi}$$

$$\bar{B}(\tau) = \frac{(1-A)\,S}{4\pi}\left(\frac{3}{4}\tau + \frac{1}{2}\right)$$

$$\frac{(l/c)T^4}{\pi} = \frac{(1-A)\,S}{4\pi}\left(\frac{3}{4}\tau + \frac{1}{2}\right)$$

$$\boxed{T^4 = \left[\frac{(1-A)\,S}{4s}\right]\left(\frac{3}{4}\tau + \frac{1}{2}\right)}$$

This is the simplest expression of the “greenhouse effect”.

Notes: 1. Sometimes $\frac{(1-A)\,S}{4s}$ is written as T_e^4 where T_e is called the <u>"effective temperature"</u> of the planet. (T_e = 254.1K for Earth).

2. In radiative equilibrium, the temperature of the surface is <u>not</u> equal to the temperature of the air immediately above the surface. In particular at z=0 (or $\tau=\tau_s$):

$$\bar{I}^{+} = \bar{B}(\tau_s) + \frac{1}{2}\bar{F}$$

$$= \overline{E_s B_s} \approx E_s \bar{B}_s$$

from previous page where $\bar{B}(\tau) = \bar{F}\left(\tfrac{3}{4}\tau + \frac{1}{2}\right)$

$$E_s \bar{B}_s = \bar{B}(\tau_s) + \frac{1}{2}\bar{F} = \bar{F}\left(\frac{3}{4}\tau_s + \frac{1}{2}\right) + \frac{1}{2}\bar{F}$$

$$= \bar{F}\left(\frac{3}{4}\tau_s + 1\right)$$

$$\boxed{T_s^4 = \left[\frac{(1-A)\,S}{4sE_s}\right]\left(\tfrac{3}{4}\tau_s + 1\right)}$$

surface temp. in radiative equilibrium

for the surface temperature in radiative equilibrium.

For the earth, let us take:

$E_s \approx 1$
A = 0.3
S = 1.35 x 10^6 erg cm^{-2} sec^{-1}
s = 5.67 x 10^{-5} erg cm^{-2} deg^{-4} sec^{-1}

$\tau \approx \tau_s \exp\left(-z/h\right)$
$\tau_s \approx 4$

h = scale ht. of principal atmospheric absorber (H_2O) $\approx$ 2 km.

And we obtain:

T_s = 359.3K
What is temp. of atmosphere near surface?
What is temperature gradient of atmosphere at surface?
What is temperature at top of atmosphere?