The Mean-Field Approximation
Baroclinic Instability

Herring (1963) originally applied "mean field theory" to convection and derived heat flux estimates which agreed pretty well with data at large Rayleigh numbers. The basic idea is that we linearize the fluctuation equations and use the resultant solutions to calculate the fluxes which alter the mean structure. These changes, in turn, affect the fluctuations. I.e., we include eddy-mean interactions, but neglect eddy-eddy ones. Herring points out that such an approximation is a form of closure neglecting third order cumulants

$$\langle a'^3 \rangle \quad , \quad \langle a'^2 b' \rangle \quad , \quad \langle a' b' c' \rangle$$

In homogeneous systems, such an approximation gives non-physical negative energies; but it is fine in inhomogeneous systems with important variations in the mean. We shall apply the theory to baroclinic instability, the major large-scale eddy transport mechanism in the mid-latitude atmosphere and certainly a significant player in the ocean as well.

Two layer model

If we have a two-layer fluid with the upper density ρ_1 and lower ρ_2, we can integrate the hydrostatic equations; writing this in terms of the dynamic pressure (pressure over density) used in the Boussinesq approximation

$$\rho_1 p_1 = \rho_1 g(h_1 + h_2 - z) \quad , \quad \rho_2 p_2 = \rho_1 g h_1 + \rho_2 g(h_2 - z)$$

so that horizontal pressure gradients are given by

$$\nabla p_1 = g\nabla(h_1 + h_2) \quad , \quad \nabla p_2 = \frac{\rho_1}{\rho_2} g\nabla h_1 + g\nabla h_2$$

or

$$\nabla p_1 = \nabla p_2 + \frac{\rho_2 - \rho_1}{\rho_2} g\nabla h_1$$

The last term is $g'\nabla h_1$ with the reduced gravity being $g' = g(\rho_2 - \rho_1)/\rho_2$.

We can now write the horizontal momentum equation

$$\frac{\partial}{\partial t}\mathbf{u} + (f_0 + \beta y + \zeta)\hat{\mathbf{z}} \times \mathbf{u} = -\nabla(p + \frac{1}{2}|\mathbf{u}|^2) + \mathcal{F} - \mu\mathbf{u}$$

the hydrostatic equation (calling $h_1 = H_1 + h$, $h_2 = H_2 - h = H - H_1 - h$ and ignoring the stretching associated with H changes)

$$p_1 = p_2 + g'h$$

and the mass conservation equations

$$\nabla \cdot \mathbf{u} - \frac{w}{H_1 + h} = 0$$

$$\nabla \cdot \mathbf{u} + \frac{w}{H_2 - h} = 0$$

with the thermodynamics at the interface represented as a transport across density surfaces

$$(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla)h + w = \mathcal{H} - \mu h$$

In terms of the transformed Eulerian mean, the flows are

$$\begin{aligned}
w^\dagger &= w + \mathbf{u} \cdot \nabla h \\
\mathbf{u}_1(H_1 + h) &= H\hat{\mathbf{z}} \times \nabla \psi_1 + H\mathbf{u}_1^\dagger \\
\mathbf{u}_2(H - h_2) &= H\hat{\mathbf{z}} \times \nabla \psi_2 + H\mathbf{u}_2^\dagger \\
p^\dagger &= p + \frac{1}{2}|\mathbf{u}|^2 - f_0\psi
\end{aligned}$$

(note that this implies

$$\nabla \cdot (H \pm h)\mathbf{u} = \nabla \cdot H\mathbf{u}^\dagger$$

—the residual flow is represents the divergent part of the mass transport).

We now get

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{u} + \hat{\mathbf{z}} \times \mathbf{u}\, q + f_0\hat{\mathbf{z}} \times \mathbf{u}^\dagger &= -\nabla p^\dagger + \mathcal{F} - \mu\mathbf{u} \\
\nabla \cdot \mathbf{u}^\dagger \mp \frac{w^\dagger}{H} &= 0 \\
\frac{\partial}{\partial t}h + w^\dagger &= \mathcal{H} - \mu h
\end{aligned}$$

with $q = \zeta + \beta y \mp f_0 h/H$ being the pseudo-potential vorticity. The hydrostatic equation becomes

$$f_0(\psi_1 - \psi_2) + \left[p_1^\dagger - p_2^\dagger + \frac{1}{2}|\mathbf{u}_1|^2 - \frac{1}{2}|\mathbf{u}_2|^2\right] = g'h$$

Advantages:

- All terms in the momentum equations are same order in Ro
- Mean cross-isopycnal flow is generated by mean heating/radiation
- Nonlinearity is limited and dominated by PV advection (though there's some in the Bernoulli function)

QG

For the QG equations, we (1) drop the term in the brackets and (2) replace $\mathbf{u}$ by $\hat{\mathbf{z}} \times \nabla\psi$ in the momentum equations and in q

$$q = \nabla^2\psi + \beta y \mp \frac{f_0^2}{g'H}(\psi_1 - \psi_2)$$

The vorticity equation becomes

$$\frac{\partial}{\partial t}(\zeta + \beta y) + \mathbf{u}\cdot\nabla q = \mp\frac{f_0 w^\dagger}{H} + \nabla\times\mathcal{F} - \mu\zeta$$

the density eqn is

$$\frac{\partial}{\partial t}f_0(\psi_1 - \psi_2) + w^\dagger = \mathcal{H} - \mu f_0(\psi_1 - \psi_2)$$

and the PV eqn.

$$\frac{\partial}{\partial t}q + \mathbf{u}\cdot\nabla q = \pm f_0\mathcal{H}/H + \nabla\times\mathcal{F} - \mu q$$

Now we separate out the zonal average and the fluctuations

$$\frac{\partial}{\partial t}q' + \frac{\partial}{\partial x}Uq' + Q_y v' + \frac{\partial}{\partial x}(u'q') + \frac{\partial}{\partial y}(v'q' - \overline{v'q'}) = -\mu q'$$

$$q' = \nabla^2\psi' \mp F(\psi_1' - \psi_2')$$

We can use a sin series for q' and ψ' since all the terms above will vanish on the walls. Here $F = f_0^2/g'H$ and $Q_y = \beta - U_{yy} \pm F(U_1 - U_2)$.

The zonal average equations are

$$\frac{\partial}{\partial t}U = \overline{v'q'} + f_0\overline{v^\dagger} + \mathcal{F} - \mu U$$

$$\frac{\partial}{\partial y}\overline{v^\dagger} \mp \frac{\overline{w^\dagger}}{H} = 0$$

$$\frac{\partial}{\partial t}\overline{h} + \overline{w^\dagger} = \mathcal{H} - \mu\overline{h}$$

Define

$$\overline{w^\dagger} \equiv \frac{f_0}{g'}\frac{\partial}{\partial y}\phi$$

so that

$$f_0\overline{v^\dagger} = \pm F\phi$$

with ϕ vanishing on the walls so that a sine expansion works for it, too. The momentum and thermal equations become

$$\frac{\partial}{\partial t}U = \overline{v'q'} + \mathcal{F} - \mu U \pm F\phi \equiv R - \mu U \pm F\phi$$

$$\frac{\partial}{\partial t}(U_1 - U_2) = \frac{\partial^2}{\partial y^2}\phi - \frac{g'}{f_0}\mathcal{H}_y - \mu(U_1 - U_2)$$

Subtracting the upper and lower momentum equations gives

$$\frac{\partial}{\partial t}(U_1 - U_2) = R_1 - R_2 - \mu(U_1 - U_2) + (F_1 + F_2)\phi$$

which, combined with the thermal equation, gives the zonal average omega equation

$$\left[\frac{\partial^2}{\partial y^2} - F_1 - F_2\right]\phi = \frac{g'}{f_0}\mathcal{H}_y + R_1 - R_2$$

with $R = \overline{v'q'} + \mathcal{F}$.

Mean field

The mean field approximation simplifies the fluctuating q' equation as

$$\frac{\partial}{\partial t}q' + \frac{\partial}{\partial x}Uq' + Q_y v' = -\mu q'$$

The zonal flow equations remain the same.

Stability

We shall arrange the forcing to produce a uniform vertical shear so that $U_j \to \mathcal{U}_j + U_j$ and scale velocities by $\mathcal{U}_1 - \mathcal{U}_2$ and lengths by the channel width and consider the case $H_1 = H_2$, $\mathcal{U}_1 = -\mathcal{U}_2$. The controlling parameters are then

$$F = F_d L^2 \quad , \quad \beta = \beta_d/\mathcal{U}_d F_d \quad , \quad \mu = \mu_d L/\mathcal{U}_d$$

Rayleigh's theorem implies a necessary condition for instability is that the gradient Q_y must change sign either in the horizontal or in the vertical; this will occur when $\beta < 1$.

Examples

Weak instability $F = 25$, $\beta = 0.5$, $\mu = 0.1$
Demos, Page 4: Examples *<F=25,beta=0.5>* *<full>* *<mean>* *<mfa>* *<comparison>*

Moderate $F = 100$, $\beta = 0.1$, $\mu = 0.1$ Demos, Page 4: Examples *<F=100,beta=0.1>* *<full>* *<mean>* *<mfa>* *<comparison>*

Strong $F = 400$, $\beta = 0$, $\mu = 0.1$ Demos, Page 4: Examples *<F=400,beta=0.0>* *<full>* *<mean>* *<mfa>* *<comparison>*

Residual circulation

We can compare the residual circulation, represented by ϕ, to the Eulerian mean meridional flow, which we can represent by

$$\overline{v^a} = \overline{v^\dagger} - \overline{v'h'}$$

or

$$\phi^a = \phi - \overline{v'(\psi_1' - \psi_2')}$$

These are for $F = 100$, $\beta = 0.1$.

Demos, Page 5: Diagnostics *<Eddy enstrophy>* *<Zonal flow>* *<mean>* *<PV gradients>* *<mean PV gradients>* *<phi>* *<phia>* *<mean>*