

Homework assignment 4

Problem 1. A Smagorinsky-type model for subgrid-scale fluxes in quasi-geostrophic turbulence

The quasi-geostrophic potential vorticity governing equation on the f -plane is,

$$\partial_t q + \mathbf{u} \cdot \nabla q - \nu \nabla^2 q = 0, \quad \mathbf{u} = (-\partial_y \psi, \partial_x \psi), \quad q = \nabla^2 \psi + \frac{\partial}{\partial z} \left(\frac{f^2}{N^2} \partial_z \psi \right). \quad (1)$$

The flow can be split into a large scale component, and a small scale component: $q = \bar{q} + q'$ and $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$, where \bar{X} represents the average value of X over the averaging scale, and $\overline{X'} = 0$.

- (a) Write an equation for the time-evolution of \bar{q} . Identify the term through which the small-scale dynamics influences the large scale vorticity. Write this term as the divergence of a flux.
- (b) Similarly write an equation for q' , multiply this equation by q' and average to obtain an equation for the small scale enstrophy $\overline{q'^2}$. Identify the terms responsible for small scale enstrophy production, dissipation and transport.
- (c) In the enstrophy cascade region of quasi-geostrophic turbulence characteristics of the flow depend only on χ , the enstrophy dissipation rate, and k , the wavenumber. Express the turbulent eddy diffusivity κ_T in terms of χ and k .
- (d) Write down an expression parameterizing the flux term in (a) in terms of diffusion down a large-scale gradient, with the eddy diffusivity κ_T .
- (e) Now assume a balance between production and dissipation of enstrophy, using the relevant terms derived in (b), to express κ_T in terms of χ and the large-scale gradients.
- (f) Eliminate χ between the expressions derived in (c) and (d) to obtain an expression for the eddy diffusivity in terms of large-scale gradients, and a lengthscale $l = k^{-1}$.

Problem 2. Rotational constraints on 3D turbulence

Over most of the ocean $N/f > 1$, so that turbulence is more likely to be influenced by stratification than by rotation. However, in regions where $N \approx 0$, such as the surface mixed layer, rotation may constrain unstratified turbulence. Assuming that rotation constrains the turbulent motions when the Rossby number $U/fL < 1$, where U is the velocity scale, and L the length scale, find the length scale ℓ above which

rotation affects the turbulence. Assume for $l < \ell$ the turbulent motions are in the inertial range of the Kolmogorov spectrum.

Problem 3. Triad interactions in two dimensions.

Consider the forced barotropic vorticity equation,

$$\partial_t q + \mathbf{u} \cdot \nabla q - \nu \nabla^2 q = F e^{i\mathbf{k} \cdot \mathbf{x}} + c.c., \quad \mathbf{u} = (-\partial_y \psi, \partial_x \psi), \quad q = \nabla^2 \psi + \beta y, \quad (2)$$

with F a complex random function of time. Examine a triad of waves

$$q \approx \sum_{j=1,2,3} q_j(t) e^{i\mathbf{k}_j \cdot \mathbf{x}}, \quad \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0, \quad (3)$$

and derive the equation for $\partial_t q_j$.

- (a) Show that interactions among the triad conserve energy and enstrophy in the absence of dissipation and forcing.
- (b) Use a similar approach to derive the equations for the energy and enstrophy in the presence of the forcing and dissipation. If the forcing is on the smallest or largest wavenumber, prove that the others will have zero amplitude in the statistically steady state.