

Homework assignment 1

Problem 1

The problem of the onset of turbulence is an area of active research. In class we briefly discussed the original model proposed by Landau and the more recent ideas of Ruelle and Takens. The textbooks by Landau and Lifshitz and Tritton have chapters on the two theories. The article "Hydrodynamic Instabilities and the Transition to Turbulence," published by H.L. Swinney and J.P. Gollub in *Physics Today* (August 1978) is another good review of the issue. Read this material (and whatever else you might find useful) and summarize the differences between Landau's explanation versus the modern view of the onset of turbulence,. Discuss what is the observational evidence which supports one or the other explanation. Your answer should not exceed a couple of pages.

Problem 2

Consider the *cubic map*,

$$x_{n+1} = r x_n(3 - 4x_n^2). \quad (1)$$

The parameter r plays the role of a Reynolds number in that it controls whether the solutions are regular or erratic. The map is defined in the interval $x \in [-1, 1]$ and $r \in [0, 1]$.

Part 1

Write a MATLAB program to draw the bifurcation diagram of the cubic map for $r \in [0, 1]$. How do the solutions change as r becomes larger? Is there a transition to a "chaotic" behavior? Does the bifurcation diagram look similar to the one of the logistic map?

Part 2

Consider the cubic map with $r = 1$. Select a set of different initial conditions x_0 and plot the PDF of this set, i.e. the $P(x_0)$. Now integrate the map for this set of initial conditions, and plot the PDF after n iterations, i.e. the $P(x_n)$. Repeat this exercise for different values of r in the range $[0, 1]$. Do the PDFs asymptote some simple distribution? How do the PDFs change as a function of r ?

Problem 3

Let x be a Gaussian variable. First show that all odd moments of x are zero. Then show that all even moments are related to the variance $\sigma^2 = \overline{(x - \bar{x})^2}$ and write what is the relationship. In particular write the skewness and the kurtosis of a Gaussian variable.

Problem 4

The logistic map presents the same closure problem encountered in turbulence. The equation for any moment of x depends on higher order moments and cannot be solved without additional information on these higher order moments. Consider for example the equation for the ensemble average of x ,

$$\bar{x}_{n+1} = r \bar{x}_n - r \overline{x_n^2}, \quad (2)$$

where the overbar denotes the ensemble average. The equation for the first moment, \bar{x} , depends on the second moment $\overline{x^2}$. Derive the equation for the second moment and shows that it depends on moments higher than second.

Most methods of closing the hierarchy of moments make assumptions about the relationship of $(n+1)$ -th order moments to n -th order moments. A typical approach is to assume that the statistics of x are Gaussian and use the results of problem 3 to relate all moments to the second. Apply this method to close the equation for the second moment. With this assumption, the equation for the first and second moment form a closed set. Use MATLAB to iterate the two equations for $r = 4$. What happens? Can you explain why this approach fails [Hint: you might want to go back to Lecture 2 and remind yourself of what is the stationary distribution for the logistic map]?