**Climate Physics and Chemistry**
Ocean and Climate: Problem Set Solution

*Comment: I paid little attention to numerical values, only looking to see if you understood the basic idea.*

1. The pressure field at depth $z = 1000$m in the ocean is found to follow the rule,

$$p = p_0 \cos\left(\pi x/10^4\right)\cos\left(\pi y/10^4\right).$$

The origin of $y$ is taken to be 30°N and $x, y$ are measured in meters. What are the northward and eastward components of geostrophic velocity at $x = 10^4/2$, $y = 0$? If the fluid density is approximated as uniform, $\rho = \rho_0 = 1.03 \times 10^3\text{kg/m}^3$, how much water (mass) is moving northward between the seasurface and 1000m, between $x = 0$ and $x = 10^4/2$? For a numerical answer, let $p_0 = 238\text{N/m}^2$.

*The general expression for northward geostrophic flow is*

$$-f\rho v = -\frac{\partial p}{\partial x} = p_0 \frac{\pi}{10^4}\sin\left(\pi x/10^4\right)\cos\left(\pi y/10^4\right),$$

*which at $x = 10^4/2$, $y = 0$ produces*

$$v = -\frac{p_0\pi}{f\rho_0 \times 10^4} = \frac{-\pi p_0}{(7.2722 \times 10^{-5})\, 1.03 \times 10^3 \times 10^4} = -4.1942 \times 10^{-3} p_0 = -1\text{m/s}$$

*To get the total mass transport, at fixed $z$,*

$$\int_0^{10^4/2} \rho_0 v dx = -\int_0^{10^4/2} \frac{p_0 \pi}{10^4 f}\sin\left(\pi x/10^4\right)\cos\left(\pi y/10^4\right) dx =$$

$$\frac{p_0}{f}\cos\left(\pi x/10^4\right)\cos\left(\pi y/10^4\right)\Big|_0^{10^4/4} = \frac{p_0}{f}(0-1) = -3.3 \times 10^6\left(\text{m}^2/\text{s}\right)\text{kg/m}^3$$

*($f$= 7.2722 × $10^{-5}$)*

To get the vertical transport, just multiply by the depth range of 1000m, as by assumption, there is no depth dependence in $v$.

2. A ship measures the temperature and salinity in the ocean at $x = 0$, and $x = 50$km at a latitude of 45° N. When converted to density, the two profiles are found to be closely approximated as,

$$\rho(x = 0, z) = 1.03 \times 10^3\text{kg/m}^3\left(1 - z/\left(2 \times 10^4\right)\right),$$
$$\rho(x = 50\text{km}, z) = 1.03 \times 10^3\text{kg/m}^3\left[1 - (z + 1 \times 10^{-7}z^2)/\left(2 \times 10^4\right)\right]$$

where $z$ is in meters. Compute and plot the northward velocity as a function of $z$ for $0 \le z \le 3000$m under the assumption that $z = -2000$m is a level of no motion. What is different at 10°N? Take gravity, $g = 10\text{m/s}^2$.

*The thermal wind relation for the flow in the northward direction,*

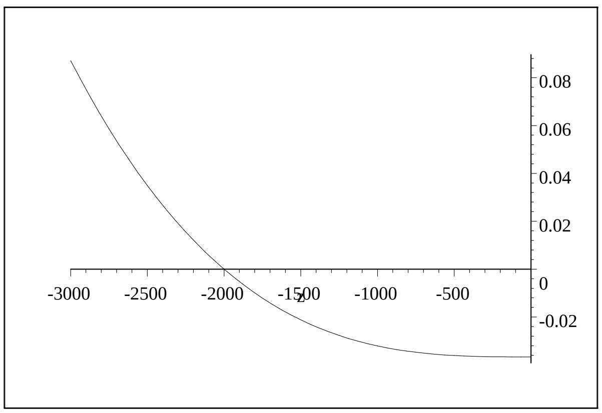
$$\rho v(z) = -\frac{g}{f}\int_{z_o}^{z}\frac{\partial\rho}{\partial x}dz + b$$

*Choose* $z_0 = -2000$ *so that* $b = 0$*. Also,*

$$\frac{\partial\rho}{\partial x} \approx \frac{\Delta\rho}{\Delta x} = \frac{1.03\times10^3\text{kg/m}\left(-1\times10^{-7}z^2\right)/\left(2\times10^4\right)}{50\times10^3\text{m}}.$$

$(\rho = 1.03\times10^3$, almost exactly. *Hence,*

$$v(z) = -\frac{g}{f\rho}\int_{-2000}^{z}\frac{1.03\times10^3\left(1\times10^{-7}z^2/\left(2\times10^4\right)\right)}{50\times10^3}dz$$
$$= -4.5835\times10^{-12}z^3 - 3.6668\times10^{-2}$$




Velocity (vertical axis) against depth (horizontal axis) in m/s.

*One can take* $\rho$ *outside the integral in the last equation, as its inclusion makes no observable difference (part of the "Boussinesq approximation"). At 10°N, the velocity will be larger in the ratio of* $\sin 30°/\sin 10°$.

3. A uniform wind blows towards the northeast such that the windstress on the ocean is $\tau = \tau_0(1,1)$. Using the equations

$$-fv = A\frac{\partial^2 u}{\partial z^2}$$
$$fu = A\frac{\partial^2 v}{\partial z^2}$$

which govern the Ekman layer, find $u, v$ as a function of $z$. (Hint: multiply the second equation by $i$ and add to the first equation. Solve this equation for the complex quantity $u + iv$.) Note that the implied density is $\rho_0 = 1$, which for seawater implies cgs units. Alternatively, one can define $A' = A/\rho$.

*This problem is worked out in almost every textbook. It can be reduced to the textbook calculations by defining new axes so that* $x'$ *and velocity* $u'$ *are oriented to the northeast, perpendicular* *to the windstress. Then $y', v'$ point towards the northwest. The equations in the new coordinate system are the same as those written above. Multiplying the second one by $i$ and adding to the first produces,*

$$-f\left(v' - iu'\right) = A\frac{\partial^2}{\partial z^2}\left(u' + iv'\right)$$

*which is the same as*

$$if\left(u' + iv'\right) = A\frac{\partial^2}{\partial z^2}\left(u' + iv'\right)$$

*an ordinary differential equation for the variable $U' = u' + iv'$*

$$\frac{d^2U'}{dz'^2} - \frac{if}{A}U' = 0, \tag{1}$$

*subject to the boundary conditions*

$$A\frac{\partial U'}{\partial z'}\bigg|_{z=0} = \sqrt{2}\tau_0$$

*(in the new coordinate system, the wind field is aligned with the with $u, x$) and that $|U'| \to 0$ as $z \to -\infty$. The square root of 2 appears because the vector $\tau_0(1,1)$ has magnitude $\sqrt{2}$ and direction in the new $x'$ sense. Differential equations like Eq. (1) are usually solved by substitution of a trial solution $U' = C\exp(pz')$ where $A, p$ are constants to be determined. Substituting this solution into (1) produces*

$$C\left(p'^2 - \frac{if}{A}\right) = 0$$

*The only way to avoid a solution $C = 0$ (which would not satisfy the boundary condition) is to have*

$$p'^2 - \frac{if}{A} = 0$$

*or,*

$$p' = \pm\sqrt{\frac{if}{A}} = \pm\sqrt{i}F = \pm\frac{\sqrt{2}}{2}\left[1 + i2\right]F$$

*So that there are two possible solutions,*

$$U' = C\exp\left(\frac{\sqrt{2}}{2}\left(\frac{f}{A}\right)^{1/2}(1+i)\right)z + D\exp\left(-\frac{\sqrt{2}}{2}\left(\frac{f}{A}\right)^{1/2}(1+i)\right)z$$

*As we go into the ocean away from the surface, $z$ becomes very large and negative. The second solution would become exponentially large, an unphysical solution for one driven by the wind at $z = 0$. So we set $D = 0$. It remains only to find $C$, and we do that by using the boundary condition,*

$$A\frac{d}{dz}\left[C\exp\left(\frac{\sqrt{2}}{2}\left(\frac{f}{A}\right)^{1/2}(1+i)\right)z\right]\bigg|_{z=0} = \sqrt{2}\tau_0$$

$$AC\frac{\sqrt{2}}{2}\left(\frac{f}{A}\right)^{1/2}(1+i) = \sqrt{2}\tau_0$$

*whose solution is,*

$$C = (1-i)\frac{\tau_0}{A\left(\frac{f}{A}\right)^{1/2}} = (1-i)\frac{\tau_0}{\sqrt{fA}}$$

*Hence, $U'$ is,*

$$U' = (1-i)\frac{\tau_0}{(fA)^{1/2}}\exp\left[\left(\frac{\sqrt{2}}{2}\left(\frac{f}{A}\right)^{1/2}(1+i)\right)z\right]$$

*The real part is $u'$, and the imaginary part is $v'$.*

$$\begin{aligned} u' + iv' &= (1-i)\frac{\tau_0}{(fA)^{1/2}}\exp\left[\left(\frac{\sqrt{2}}{2}\left(\frac{f}{A}\right)^{1/2}(1+i)\right)z\right] \\ &= \frac{\tau_0 e^{\frac{\sqrt{2}}{2}\left(\frac{f}{A}\right)^{1/2}z}}{(fA)^{1/2}}\left\{\left[\cos\frac{\sqrt{2}}{2}\left(\frac{f}{A}\right)^{1/2}z + \sin\frac{\sqrt{2}}{2}\left(\frac{f}{A}\right)^{1/2}z\right] + i\left[-\cos\frac{\sqrt{2}}{2}\left(\frac{f}{A}\right)^{1/2}z + \sin\frac{\sqrt{2}}{2}\left(\frac{f}{A}\right)^{1/2}z\right]\right\} \end{aligned}$$

*At the surface, $u' = \tau_0/(fA)^{1/2}$ , $v' = -\tau_0/(fA)^{1/2}$, that is, 45° to the right of the wind. As $z'$ decreases (becomes negative) below the seasurface, a diminishing spiral is seen.*