Quantifying Uncertainty

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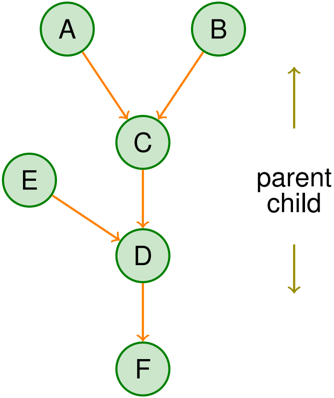
Revisiting Graphs and Hierarchical Bayes:



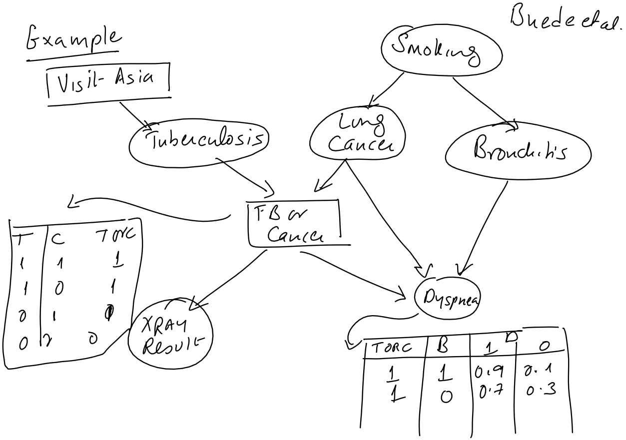
a Hierarchical relationship between variables

b All are random

c Represented by directed acyclic graphs

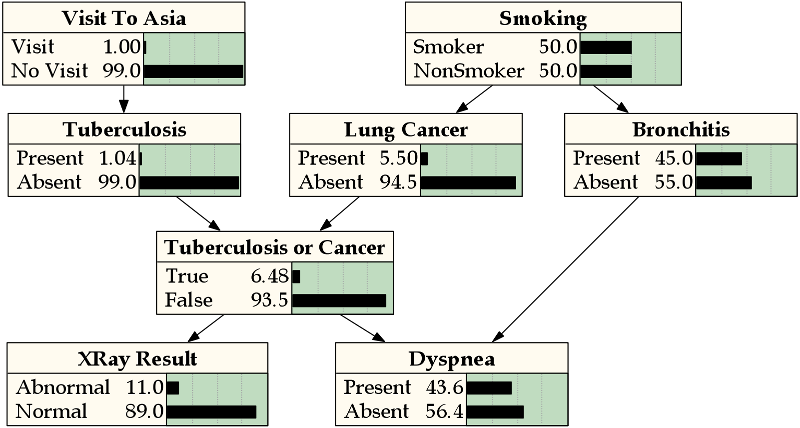
$\Rightarrow$ Bayesian Networks

example



Buede et al.

example


Visit To Asia
Visit 1.00
No Visit 99.0
Smoking
Smoker 50.0
NonSmoker 50.0
Tuberculosis
Present 1.04
Absent 99.0
Lung Cancer
Present 5.50
Absent 94.5
Bronchitis
Present 45.0
Absent 55.0
Tuberculosis or Cancer
True 6.48
False 93.5
XRay Result
Abnormal 11.0
Normal 89.0
Dyspnea
Present 43.6
Absent 56.4

Bayesian Networks can be used to model a large “ interdisciplinary” dependences and assess

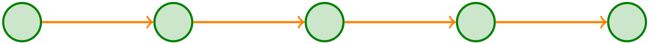
a Evidence

b Uncertainty

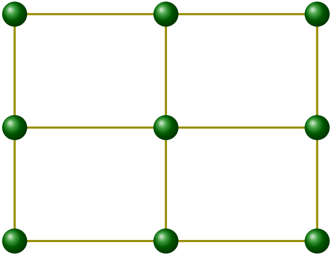
You will find tons of material and papers using Bayesian Networks. Especially in Ecological and Environmental application. and climate

Markov Networks

A markov chain is a Bayesian Network

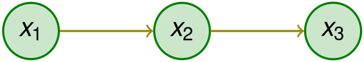


We may model “lattices” through Markov Networks



Markov random field example “ two-way interactions”

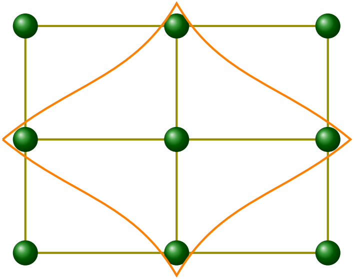
Recall again



$$
\begin{aligned}
P(x_1, x_2, x_3) &= p(x_1|x_2, x_3)p(x_2|x_3)p(x_3) \\
&= p(x_1|x_2)p(x_2|x_3)p(x_3) \\
&= p(x_3|x_2)p(x_2|x_1)p(x_1) \\
&= p(x_1|x_2)p(x_3|x_2)p(x_2)
\end{aligned}
$$

General Formulation

MRF==Gibbs Random Fields



Clique$\equiv N_c(X_{ij})$

$$P(X_{ij}|\underline{X}) = P(X_{ij}|N_c(X_{ij}))$$

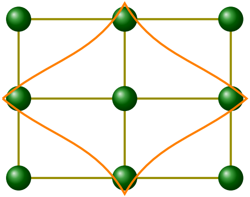
E.g Ising model for feromagnetism

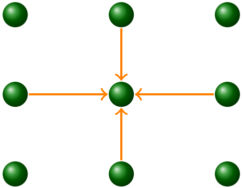
$X_{i,j} \in \{-1, 1\}$

$$E_{ij}(x) = \frac{-1}{kT} \sum_{X_{l,m} \in N_c(X_{ij})} X_{ij} \cdot X_{lm}$$

$$P(x) = \frac{1}{Z} e^{-\sum_{ij} E_{ij}(x)}$$

The joint and conditional views





$$P(x) = \prod_{ij} \psi_{ij}(x_{ij}, N_c(x_{ij}))$$

$$P(x_i | \underline{x} \backslash x(i,j)) = p(x_{ij} | N(x_{ij}))$$

Hammersley-Clifford

$$P(\underline{x}) = \frac{1}{Z} \underset{c \in CL(g)}{} \phi_c(x_c)$$

c-clique, $\forall x \in c$, $C \subseteq \{x, N(x)\}$

For Bayesian Networks

$$p(x) = \prod_{i} p(x_i | pa(x_i))$$

pa -parent

For Markov Networks

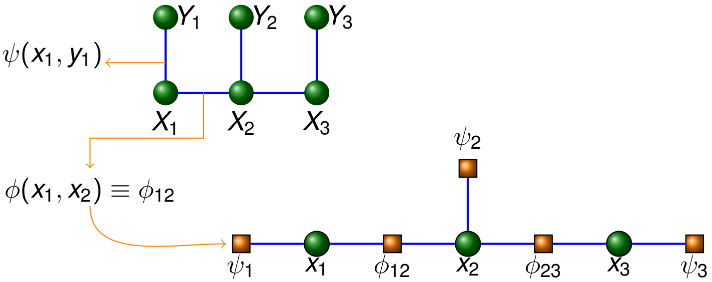
$$p(x) = \prod_{i} p(x_i) \prod_{ij} p(x_i, x_j)$$

Probabilities and Potentials

$$P(x) \propto \underset{i}{\quad} \psi(x_i) \underset{ij}{\quad} \phi(x_i, x_j)$$

So“renormalization” doesn’t become an issue

Factor graph



Inference and Belief Propagation

What is $p(\underline{x}) \equiv p(x_1, x_2, \ldots, x_n)$?
what is $p(x_1)$?

$$
\begin{aligned}
p(x_1) &= \sum_{x_3} \sum_{x_2} p(x_1, x_2, x_3) \\
&= \sum_{x_3} \sum_{x_2} \phi(x_1)\psi(x_1, x_2)\phi(x_2)\psi(x_2, x_3)\phi(x_3) \\
&= \phi(x_1) \sum_{x_2} \psi(x_1, x_2)\phi(x_2) \sum_{x_3} \phi(x_3)\psi(x_2, x_3)
\end{aligned}
$$

Message Passing

$$\phi(x_1) \underset{x_2}{\quad} \psi(x_1, x_2)\phi(x_2) \underbrace{\underset{x_3}{\quad} \psi(x_2, x_3)\phi(x_3)}_{\mu_{3\to 2}(x_2)}$$

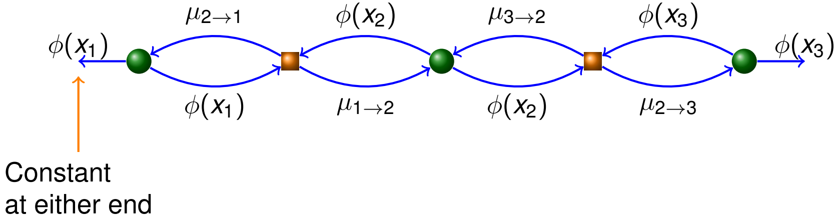
$$\mu_{2\to 1}(x_1) = \underset{x_2}{\quad} \psi(x_1, x_2)\phi(x_1)\psi_{3\to 2}(x_1)$$

Generalizing

$$b_j(x_j) = \underset{k \in N_c(j)}{} \mu_{k\to j}(x_j)$$

$$\mu_{j\to i} \quad \mu_i^j(x_i) = \underset{j}{} \psi(x_i, x_j) \underset{k \in N_c(j)}{} \mu_{k\to j}(x_j)$$

Example



"Forward-Backward","Frontier Propagation": variety of schemes

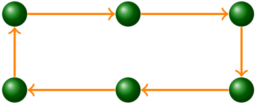
Inference on

- Markov Networks
- Bayesian(Belief) Networks

Via

- Belief Propagation
- As an example, let's look at EnKF/S from previous lecture as message passing.

Limitations



- Graphs with loops: BP does not converge globally.

Why?

$$\phi(x_1) \underset{x_2}{} \psi(x_1, x_2)\phi(x_2) \underset{x_3}{} \psi(x_2, x_3)\phi(x_3)\psi(x_1, x_3)$$

Have to carry x_1 around
But-local convergence is often “good enough”