14.01 Problem Set 2

SOLUTIONS

October 6, 2023

1 Income and Substitution Effects [10 points]

Ben consumes only apples (x) and t-shirts (y). His preferences can be represented by the following utility function: $U(x, y) = x^2y^3$ The price of apples is p_x, the price of t-shirts is p_y, and Ben has an income of m dollars. Derive Ben's demand for apples and t-shirts as a funtion of p_x, p_y and m. Suppose that initially the prices are $p_x = p_y = 1$ and income is $m = 5$. How many apples does Ben buy? Now suppose that the price of apples increases to $p_x = 2$. How many apples will he buy now? How much of the drop in demand for apples is due to the substitution effect and how much is due to the income effect? Calculate this numerically and show it in a graph.

Solution: Ben's demand for apples and t-shirts as a funtion of p_x, p_y and m is

$$x = \frac{2}{5}\frac{m}{p_x} \qquad y = \frac{3}{5}\frac{m}{p_y}$$

Initially $x = 2$ and $y = 3$, so the initial level of utility is $\bar{u} = 108$. After the change in prices, $x = 1$ and $y = 3$. To find the income needed to attain the level of utiilty $\bar{u}$ at the new prices we use the following system:

$$MRS = -\frac{p'_x}{p_y} \qquad u(x, y) = \bar{u}$$

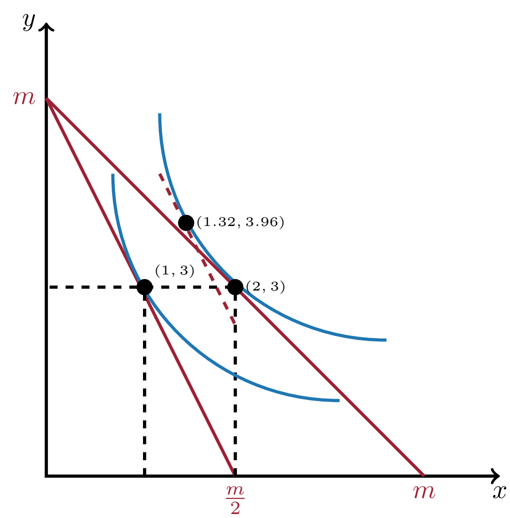
Replacing with our values

$$\frac{2}{3}\frac{y}{x} = 2 \qquad x^2y^3 = 108$$

From the first equation we have $y = 3x$. In the second equation

$$27x^5 = 108 \implies x \approx 1.32 \qquad y \approx 3.96$$

Then, the drop in the demand for apples due to the substitution effect is equal to $2 - 1.32 = 0.68$, and the drop due to the income effect is equal to $1.32 - 1 = 0.32$.



2 True or False (15 Points)

For each of the following statements, indicate if they are True or False. Justify your answer.

1. *(3 Points)* The short run average cost is always at least as large as the long run average cost (That is, if x is the short run cost and y is the long run cost then $x \geq y$).

   Solution: True. The short run cost is higher than or equal to the long-run cost because the producer can optimize all inputs to minimize production costs. In other words, the short-run cost is always attainable in the long run, but the firm has more degrees of freedom to minimize costs in the long run.

2. *(3 Points)* A firm with constant returns to scale in the short run must also have constant returns to scale in the long run.

   Solution: False. In the short run, at least one factor of production, such as capital, is fixed. Under these conditions, the firm may display constant returns to scale. However, in the long run, all factors are variable, which could result in varying returns to scale rather than constant ones.

3. *(3 Points)* In a competitive market, firms maximize profits by choosing the price that equals their marginal cost.

   Solution: False. In a perfectly competitive market, firms don't choose the price; they take it as given. They maximize profits by adjusting output so that the market price equals their marginal cost.

4. *(3 Points)* If a firm's production function exhibits constant returns to scale then it has diminishing marginal returns on every factor of production.

   Solution: False. Consider the case of perfect substitution, where the production function is represented by $F(L, K) = L + K$. This function exhibits constant returns to scale, but both factors also have constant marginal returns.

5. *(3 Points)* A firm with increasing returns to scale has decreasing marginal costs.

   Solution: True. As production increases, firms become more efficient, leading to a decrease in marginal costs.

3 Returns to scale (20 Points)

Consider a firm with the following production function

$$F(K, L) = L^\alpha K^\beta$$

where α and β are real numbers between 0 and 1.

1. *(3 Points)* How does the returns to scale of the firm's production function depend on α and β?

   Solution: Suppose the firm is scaled up by a, then the production function becomes $F(aK, aL) = a^{\alpha+\beta} L^\alpha K^\beta$. The firm will have increasing returns to scale if $\alpha+\beta > 1$, decreasing returns to scale if $\alpha+\beta < 1$, and constant returns to scale if $\alpha + \beta = 1$.

2. *(2 Points)* Find the marginal product of labor and capital and the marginal rate of technical substitution between capital and labor.

   Solution:

$$MP_L = \frac{\partial F}{\partial L} = \alpha L^{\alpha-1} K^\beta$$

$$MP_K = \frac{\partial F}{\partial K} = \beta L^\alpha K^{\beta-1}$$

$$MRTS = -\frac{MP_L}{MP_K} = -\frac{\alpha K}{\beta L}$$

Take as given that the long run total cost curve for this production function is

$$c(q) = (\alpha + \beta) \left(\frac{w}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{r}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}}$$

3. *(5 Points)* Derive the long run marginal cost curve and the average cost curve for this producion function.

   Solution:

$$MC(q) = \frac{\partial c}{\partial q} = \left(\frac{w}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{r}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}-1}$$

$$AVC(q) = \frac{c(q)}{q} = (\alpha + \beta) \left(\frac{w}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{r}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}-1}$$

4. *(10 Points)* Plot and compare the long run marginal and average cost curves when $\alpha + \beta < 1$, $\alpha + \beta = 1$ and $\alpha + \beta > 1$. Your graph does not need to be to scale but should be qualitatively accurate in terms of comparing the marginal and average cost curves, and their dependence on q. Provide an intuition, using the concept of returns to scale.

   Solution: Marginal returns to scale pose a relationship between the increase in inputs and the increase in output. In particular, whether the increase in output is more than one-to-one, less than one-to-one or one-to-one with respect to an increase in output. The first case corresponds to increasing returns to scale, the second case to decreasing returns to scale, and the third case to constant. As a result, this generates the following relationship between average variable cost and marginal costs:

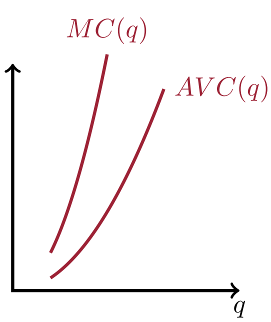


Figure 1: $\alpha + \beta < 1$

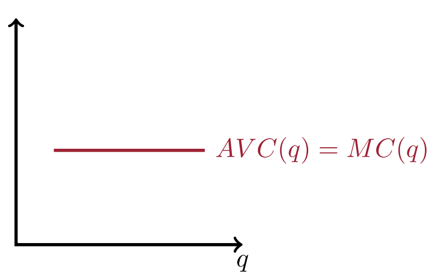


Figure 2: $\alpha + \beta = 1$

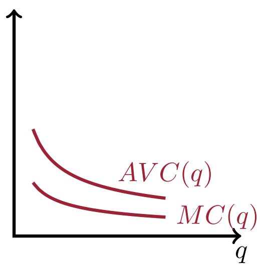


Figure 3: $\alpha + \beta > 1$

4 Short Run and Long Run Cost (30 Points)

Consider a firm with the following production function

$$F(L,K) = L^{\frac{1}{3}}K^{\frac{1}{3}}$$

where L denotes labor and K denotes capital.

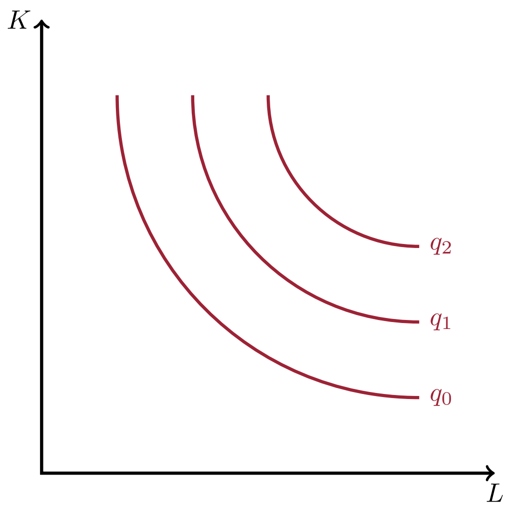
1. *(5 Points)* Find the marginal product of labor and capital and the marginal rate of technical substitution for the firm.

   Solution:

$$MP_L = \frac{\partial F}{\partial L} = \frac{L^{-\frac{2}{3}}K^{\frac{1}{3}}}{3}$$
$$MP_K = \frac{\partial F}{\partial K} = \frac{L^{\frac{1}{3}}K^{-\frac{2}{3}}}{3}$$
$$MRTS = -\frac{MP_L}{MP_K} = -\frac{K}{L}$$

2. *(5 Points)* Graph and label the isoquant map.

   Solution:



where $q_0 < q_1 < q_2$;

Suppose the wage is $w = 1$ and the cost of capital is $r = 3$. Assume that in the short run, the firm has fixed capital at $\bar{K} = 1$.

3. *(5 Points)* Derive the firm's short-run total cost curve as a function of quantity q.

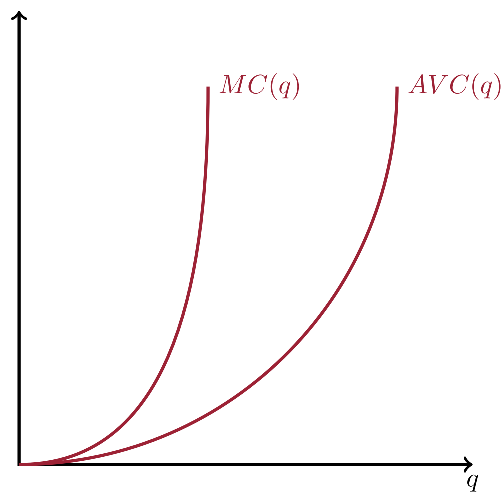
   Solution: If $\bar{K} = 1$ then $L = q^3$ and the total cost curve is equal to

$$c(q) = 3 + q^3$$

4. *(5 Points)* Derive the short-run marginal cost and average variable cost curves and plot them. Your graph does not need to be to scale but should be qualitatively accurate in terms of shapes and points of intersection, etc.

   Solution:

$$MC(q) = 3q^2$$
$$AVC(q) = q^2$$



5. *(5 Points)* What is the firm's short-run supply curve? Over what prices will the firm produce a positive quantity?

   Solution: The firm's short-run supply curve is

$$p = MC(q) = 3q^2$$

   Notice that since $p = MC(q) > AV(q)$ then the firm always chooses to produce at any price.

6. *(5 Points)* As we showed in Question 3, the total long run cost curve for a production function of the type used in this question is

$$c(q) = (\alpha + \beta)\left(\frac{w}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{r}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} q^{\frac{1}{\alpha+\beta}}$$

   Derive the total long run cost curve for $F(K, L) = L^{\frac{1}{3}}K^{\frac{1}{3}}$. Is the total cost curve in the long run larger than, smaller than or equal to the total cost curve in the short run when $\bar{K} = 1$?

   Solution: Here $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{3}$ then the total long run cost curve is

$$c(q) = \frac{2}{3}(3w)^{\frac{1}{2}}(3r)^{\frac{1}{2}}q^{\frac{3}{2}} = 2w^{\frac{1}{2}}r^{\frac{1}{2}}q^{\frac{3}{2}}$$

   The long run cost curve is smaller than or equal to the short run curve for all points because the firm is choosing its inputs optimally, whereas in the short run the level of capital is given. It is strictly smaller unless w, r, and q are such that the long run demand for capital is $K = 1$.

   Note that you can demonstrate this mathematically by comparing $2w^{\frac{1}{2}}r^{\frac{1}{2}}q^{\frac{3}{2}}$ and $wq^3 + r$, which we obtain

$$2w^{\frac{1}{2}}r^{\frac{1}{2}}q^{\frac{3}{2}} \leq wq^3 + r \iff 0 \leq (w^{\frac{1}{2}}q^{\frac{3}{2}} - r^{\frac{1}{2}})^2.$$

   The inequality holds when $\frac{w}{r} = \frac{1}{q^3} = \frac{1}{LK}$. By equating the $MRTS$ with $-w/r$, we get $\frac{w}{r} = \frac{K}{L}$. This implies $\frac{K}{L} = \frac{1}{LK}$, which yields $K = 1$ as desired.

5 Short Run Cost and Changes in Productivity (25 Points)

Consider a firm with the following production function

$$F(L, K) = AL^{\frac{1}{3}}K^{\frac{1}{3}}$$

where A is an exogenous variable that represents technology.

1. *(5 Points)* Find the marginal product of labor and capital, and the marginal rate of technical substitution for the firm. How does an increase technological improvement, represented by an increase in A, affect the the marginal product of labor, the marginal product of capital and the $MRTS$?

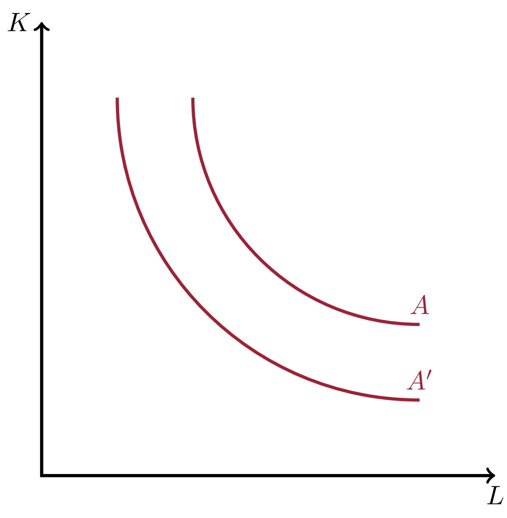
   Solution:

$$MP_L = \frac{\partial F}{\partial L} = \frac{AL^{-\frac{2}{3}}K^{\frac{1}{3}}}{3}$$
$$MP_K = \frac{\partial F}{\partial K} = \frac{AL^{\frac{1}{3}}K^{-\frac{2}{3}}}{3}$$
$$MRTS = -\frac{MP_L}{MP_K} = -\frac{K}{L}$$

   Technological improvements will equally increase both the marginal product of labor and capital, while the marginal rate of technical substitution for the firm remains constant.

2. *(5 Points)* Draw and compare the isoquants for firm production $q = 30$ for $A = 1$ and $A' = 2$. Explain the intuition behind any changes in these isoquants.

   Solution: The isoquants shift inward towards both the x axis and the y axis, indicating that the same output can be achieved with fewer inputs of K and L. This is due to the technological improvement from $A = 1$ to $A' = 2$. Over time, if technology continues to improve, the isoquants will move even closer to the origin. Intuitively, this means that advancing technology allows for increasingly efficient use of resources to produce the same output level. Notice that technological progress, represented by an increase in A, affects both capital and output equally.



Now consider a firm with the following production function:

$$F(K, L) = (L + A)^{\frac{1}{3}} K^{\frac{1}{3}}$$

3. *(5 Points)* Find the marginal product of labor and capital, and the marginal rate of technical substitution for the firm. How does an increase technological improvement, represented by an increase in A, affect the the marginal product of labor and capital and the $MRTS$? Compare with your result from 1.

   Solution:

$$MP_L = \frac{\partial F}{\partial L} = \frac{(L + A)^{-\frac{2}{3}}K^{\frac{1}{3}}}{3}$$

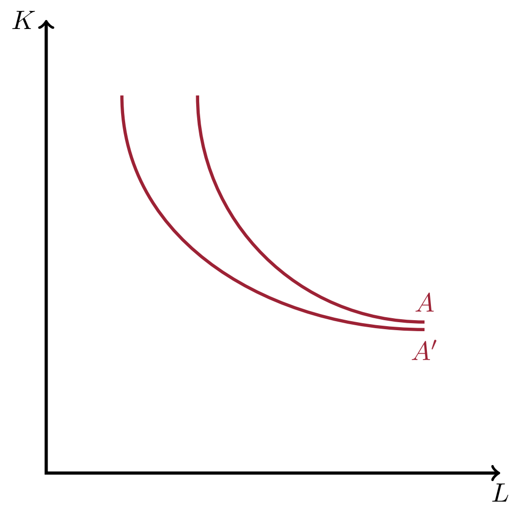
$$MP_K = \frac{\partial F}{\partial K} = \frac{(L+A)^{\frac{1}{3}} K^{-\frac{2}{3}}}{3}$$

$$MRTS = -\frac{MP_L}{MP_K} = -\frac{K}{L+A}$$

Technological improvements will increase the marginal product of both labor and capital, while the marginal rate of technical substitution for the firm is decreased. This is different from part 1, as the MRTS now depends on technology. In particular, as A increases the firm is less willing to substitute capital for labor.

4. *(5 Points)* Draw and compare the isoquants for firm production $q = 30$ for $A = 1$ and $A' = 2$. Explain the intuition behind any changes in these isoquants.

Solution: The isoquants shift to the left by 1 unit, indicating that the same output can now be achieved with one fewer unit of L while keeping K constant. This shift is due to the technological improvement from $A = 1$ to $A' = 2$. Over time, if technology continues to advance, the isoquants will move even closer to the origin. Intuitively, this suggests that progressing technology allows for increasingly efficient resource use to maintain the same output level but its impact on labor is higher than the impact on capital.



5. *(5 Points)* Which of the two production functions shown above is more compatible with the following situation: A car manufacturer can use robots to replace workers in the workplace. Justify your answer.

Solution: The second one. In this case, robots is a substitute for workers so it is additive with respect to labor.