# Lecture 5 - The Expenditure Function, with an Application to Gift Giving

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# 1 The Expenditure Function

So far, we've analyzed problems where income was held constant and prices changed. This gave us the Indirect Utility Function. Now, we want to analyze problems where utility is held constant and expenditures change. This gives us the Expenditure Function.

These two problems are closely related—in fact, they are *'duals.'* Most economic problems have a *dual problem*, which means an inverse problem. For example, the dual of choosing output in order to maximize profits is minimizing costs at a given output level; cost minimization is the dual of profit maximization. Similarly, the dual of maximizing utility subject to a budget constraint is the problem of minimizing expenditures subject to a utility constraint. Minimizing costs subject to a minimum utility constraint is the dual of maximizing utility subject to a (maximum) budget constraint.

## 1.1 Setup of expenditure function

Consumer's primal problem: maximize utility subject to a budget constraint. Consumer's dual problem: minimizing expenditure subject to a utility constraint (i.e. a level of utility the consumer must achieve). The dual problem yields the "expenditure function," the minimum expenditure required to attain a given utility level.

1. Start with:

$$\begin{aligned} & \max U(x, y) \\ s.t.\ p_x x + p_y y \quad \leq \quad & I \end{aligned}$$

2. Solve for $x^*$, $y^* \Rightarrow u^* = U(x^*,\, y^*)$ given $p_x, p_y, I$.
   $V = V\left(p_x, p_y, I\right)$

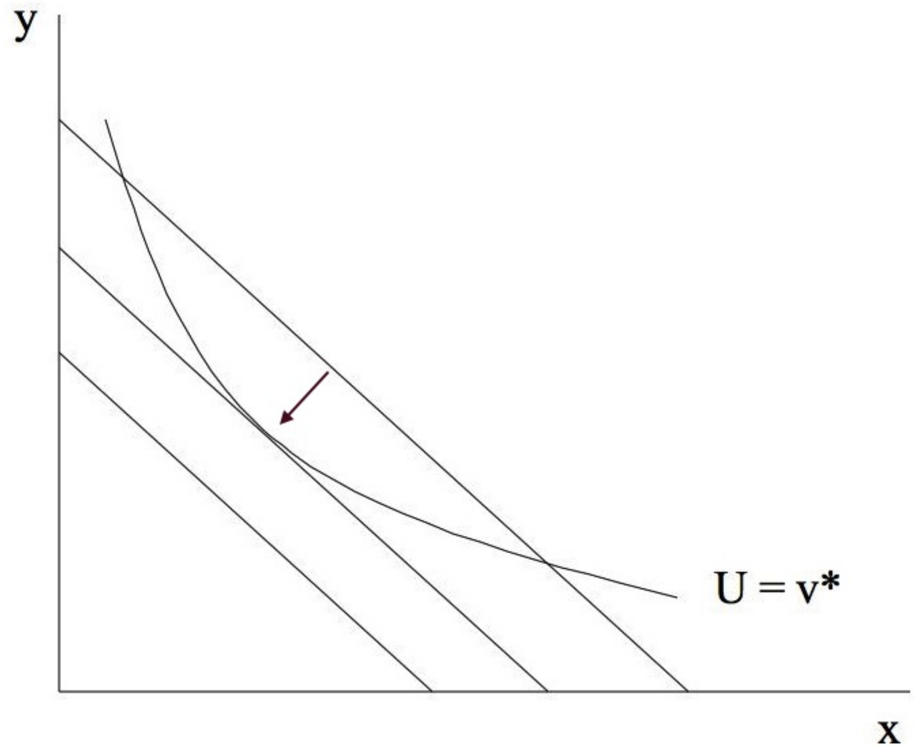
   $V$ is the indirect utility function, and its solution is equal to $u^*$

3. Now solve the following problem:

$$\begin{aligned} & \min p_x x + p_y y \\ s.t. U(x, y) \quad \geq \quad & u^* \end{aligned}$$

   which gives $E = p_x x^* + p_y y^*$ for $U(x^*,\, y^*) = u^*$.
   $E = E\left(p_x, p_y, V^*\right)$
   $E$ is the indirect utility function and its solution is equal to $p_x x^* + p_y y^*$

### 1.2 Graphical representation of the dual problem




- The dual problem consists of choosing the lowest budget set tangent to a given indifference curve. Example:

$$\begin{aligned} \min E &= p_x x + p_y y \\ s.t.\ x^{.5} y^{.5} &\geq U_p \end{aligned}$$

where $U_p$ comes from the primal problem.

$$\mathcal{L} = p_x x + p_y y + \lambda \left( U_p - x^{.5} y^{.5} \right)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= p_x - \lambda .5 x^{-.5} y^{.5} = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= p_y - \lambda .5 x^{.5} y^{-.5} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= U_p - x^{.5} y^{.5} = 0 \end{aligned}$$

- The first two of these equations simplify to:

$$x = \frac{p_y y}{p_x}$$

We substitute into the constraint $U_p = x^{.5}y^{.5}$ to get

$$\begin{aligned} U_p &= \left(\frac{p_y y}{p_x}\right)^{.5} y^{.5} \\ x^* &= \left(\frac{p_y}{p_x}\right)^{.5} U_p,\; y^* = \left(\frac{p_x}{p_y}\right)^{.5} U_p \\ E^* &= p_x \left(\frac{p_y}{p_x}\right)^{.5} U_p + p_y \left(\frac{p_x}{p_y}\right)^{.5} U_p \\ &= 2p_x^{.5} p_y^{.5} U_p \end{aligned}$$

### 1.3 Relationship between the Expenditure function and the Indirect Utility function

How do the solutions to the Dual and Primal problems compare?

- Examining the relationship between the expenditure and indirect utility functions:

$$\begin{aligned} V(p_x, p_y, I_0) &= U_0 \\ E(p_x, p_y, U_0) &= I_0 \\ V(p_x, p_y, E(p_x, p_y, U_0)) &= U_0 \\ E(p_x, p_y, V(p_x, p_y, I_0)) &= I_0 \end{aligned}$$

- The Expenditure function and Indirect Utility function are *inverses* one of the other.
- Let's verify this in the example we saw above. Recall that the primal problem gave us factor demands $x_p^*,\ y_p^*$ as a function of prices and income (not utility).
- The dual problem gave us expenditures (budget requirement) as a function of utility and prices.

$$x_p^* = \frac{I}{2p_x},\; y_p^* = \frac{I}{2p_y},\; U^* = \left(\frac{I}{2p_x}\right)^{.5} \left(\frac{I}{2p_y}\right)^{.5}$$

Now plug these into expenditure function:

$$E^* = 2U_p p_x^{.5} p_y^{.5} = 2\left(\frac{I}{2p_x}\right)^{.5} \left(\frac{I}{2p_y}\right)^{.5} p_x^{.5} p_y^{.5} = I$$

Finally notice that the multipliers are such that the multiplier in the dual problem is the inverse of the multiplier in the primal problem.

$$\begin{aligned} \lambda_P &= \frac{U_x}{p_x} = \frac{U_y}{p_y} \\ \lambda_D &= \frac{p_x}{U_x} = \frac{p_y}{U_y} \end{aligned}$$

## 1.4 Expenditure function: What is it good for?

The expenditure function is an essential tool for making consumer theory operational for public policy analysis. Using the expenditure function, we can 'monetize' otherwise incommensurate trade-offs to evaluate costs and benefits. The need for this type of calculation arises frequently in policy analysis and is the basis for most cost-benefit analyses.

As we have stressed earlier the semester, we don't know that 'utils' are. This presents a problem if we want to determine *how much* harm or benefit a certain policy imposes on an individual. The expenditure function gives us a convenient way to potentially circumvent this problem. Using the expenditure function, we can figure out *how much money* a consumer would have to be compensated (which could be a positive or negative number) to leave her equally well off after a policy is implemented as she was initially. So, the expenditure function permits us to calculate a 'money metric.'

Let's say we were considering a policy that raised prices for some consumers, perhaps by raising the cost of gasoline. Policymakers might be legitimately concerned that this policy change would adversely affect low income consumers. To offset this effect, they might provide cash compensation to offset their loss. How large should this transfer be?

A typical policy response would be to set compensation equal to the full amount of the price increase multiplied by the consumer's initial expenditure on gasoline. Let $C$ equal the compensation amount, with

$$C = \Delta P_g \times Q_{g,0}.$$

Here, $\Delta P_g = P_{g,1} - P_{g,0}$ is the policy-induced price change and $Q_{g,0}$ is the quantity that the consumer was purchasing initially (i.e., at time $t = 0$).

Is $C$ the right amount of compensation—this is, neither too much or too little? If you knew the consumer's utility function (a tall order, of course), you could calculate the exact answer as

$$C^* = E\left(P_{g1}, P_a, V\left(P_{g0}, P_a, I_0\right)\right) - E\left(P_{g0}, P_a, V\left(P_{g0}, P_a, I_0\right)\right),$$

where $I_0$ is the consumer's initial budget, $P_a$ are the prices of all other goods (assumed constant over time), and $V(\cdot)$ is the indirect utility function. You would then directly compare $C^* \lesseqgtr C$ to see if $C$ is above or below the exact compensation required. Absent knowledge of each consumer's utility function, can we say anything more?

The answer is yes. A bit of thought should convince you that it must be the case that

$$C \geq C^*.$$

That is, the simple compensation scheme $C = \Delta P \times Q_0$ *always* weakly overestimates the actual compensation required. Why? As a starting point, note that $C$ *must* be an upper bound on $C^*$. Clearly, if we compensate the consumer the full amount of money required to buy her initial bundle, she must be at least as well off as before; she can have the original bundle *or* she can choose many others that were not initially affordable. (These other bundles would have less gasoline but more of other goods else. You should demonstrate to yourself that some previously infeasible bundles are now feasible with prices $P_{g1}, P_a$ and income $I_0 + C$.)

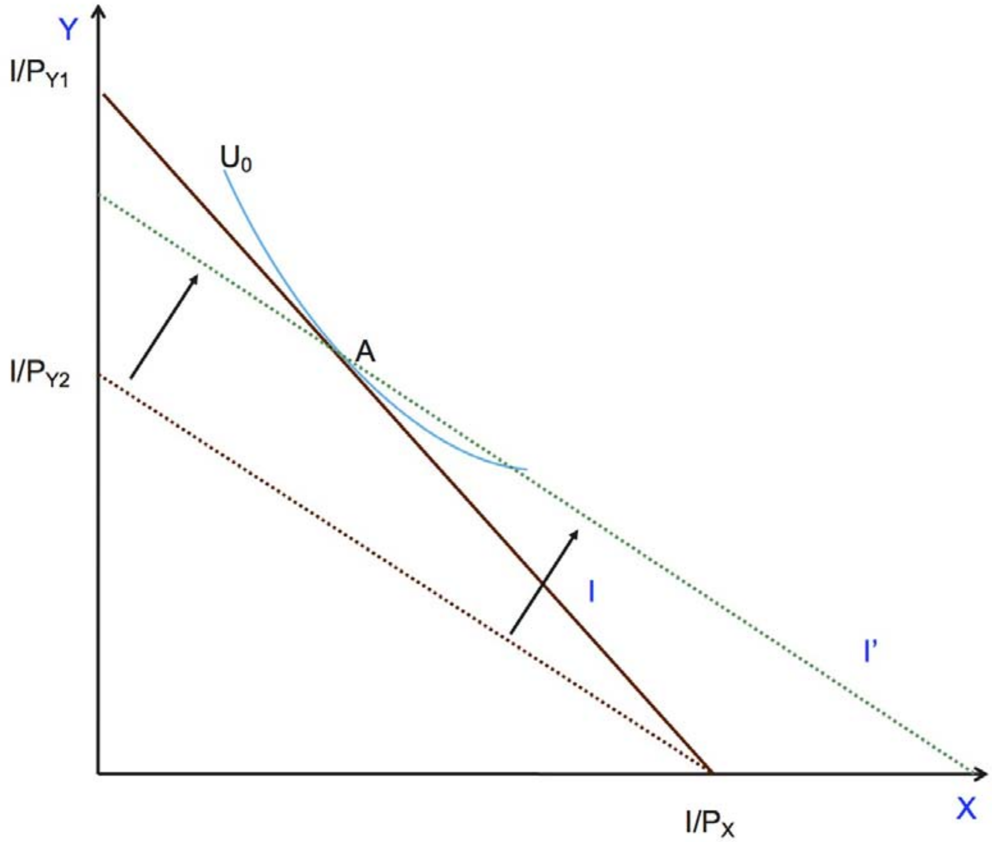
But we can say more? Again, yes. If the consumer has standard indifference curves that are bowed towards the origin (diminishing MRS), a change in the price of gasoline will cause the consumer to partially substitute towards other goods. This substitution partly blunts the effect of the price increase as the consumer re-optimizes her bundle given the new prices. If we raise the price of gasoline but hold the consumer's utility at its initial level, her optimally chosen bundle will rotate along the original indifference curve to a new location where the new price ratio is tangent to the initial indifference curve. This new bundle will cost more at the new prices than the original bundle at the old prices (unless there exists a perfect substitute for gasoline available at the original price[1]). But this new bundle (which holds utility constant) will cost strictly less than $I_0 + C$. The difference between the cost of the new bundle at the new prices and the cost of the old bundle at the old prices (both lying on the initial indifference curve) is equal to $C^*$.

### Compensation and Over-Compensation: Graphical Illustration

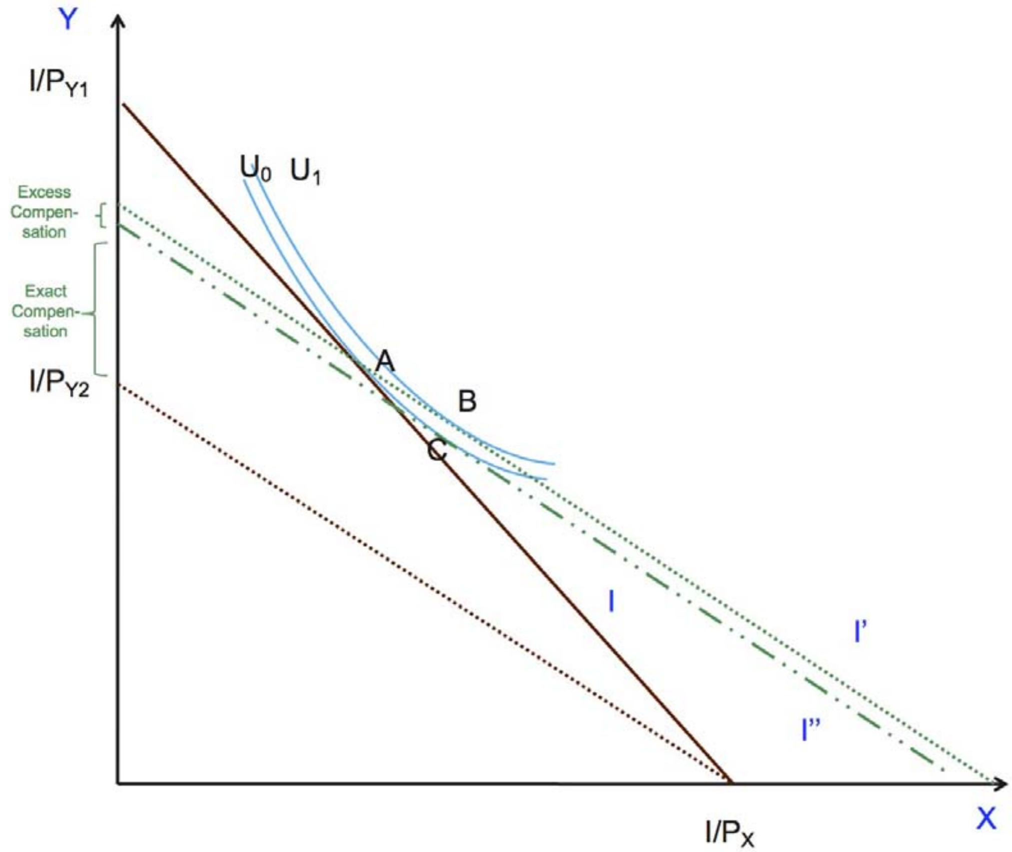
You can see the operation of this logic in the two following two figures. In the first figure, the consumer has budget set $I$ and faces prices $(P_X, P_{Y1})$ and chooses point $A$ on the budget set, generating utility $U_o$. Notice that the points of intersection of the budget set with the axes correspond to $I/P_X$ and $I/P_{Y1}$; these are the points at which the entire budget is spent on one good or the other.

Imagine the price of $Y$ rises from $P_{Y1}$ to $P_{Y2}$, rotating the budget set counterclockwise from the $x$-intercept so that the new point of intersection with the $y$-axis is given by $I/P_{Y2}$. Clearly, the consumer can no longer afford bundle $A$. How much compensation (in budget terms) do we need to give the consumer to make her as well off as she was initially at point $A$?

[1] Consider an extreme case where gasoline and kerosene are *perfect* substitutes and have identical initial prices. In that case, a rise in the price of gasoline would have no effect on consumer welfare since consumers would simply switch to kerosene. If consumers received compensation $C$ along with the price change, they would be strictly better off than before the price change.




An intuitive answer is that we would need to increase her budget from $I$ to $I'$, thus permitting the consumer to afford her old bundle at the new prices. But if you stare at the diagram for a moment, you can see that this is *excess* compensation. On the budget set $I'$, the agent can certainly afford $A$, but that budget will not be tangent to the original indifference curve; indeed it cuts right through it!




Facing budget set $I'$, the consumer would select a point like $B$ that is on a higher indifference curve. How do we know it will be on a higher indifference curve? So long as the consumer has preferences that are at least weakly convex, she will adjust her bundle as relative prices change, with the degree of adjustment depending on the curvature of her indifference curves. If her indifference curves were linear (implying that $X$ and $Y$ are perfect substitutes), the agent would simply consume exclusively $Y$ when the price of $X$ rose and the price change would have no impact on her utility. With conventional strongly convex indifference curves as pictured above, the consumer would substitute towards $Y$ as $P_X$ rises, but she would not substitute all the way.

So, how much would we need to compensate the agent to make her as well off as she was at $A$? We need to find the minimum expenditure that allows her to reach utility level $U_0$ at the new prices. This budget corresponds to $I''$. Facing that budget, the consumer would pick point $C$, which lies on the same indifference curve as $A$. The amount of compensation required is depicted in terms of the good $Y$. The additional income required is the gap between the y-intercept of $I$ and $I''$ multiplied by $P_{Y2}$. (This converts quantities of $Y$ to dollars. You could of course do the same calculation in terms of $X$.) If we had compensated the consumer the difference between $I$ and $I'$, she would be overcompensated.

This figure illustrates a key principle: the expenditure function optimizes quantities in response to prices. If consumers have convex preferences, the expenditure adjustment required to fully offset a large price change is less than the initial quantity consumed times the change in price. (For an infinitesimal price change, however, the envelope theorem applies; the consumer would not re-optimize quantity for an infinitesimal price change.)

### 1.5 Another example: Energy efficiency subsidies

Example: In 2007, the U.S. federal government enacted a policy that is slowly phasing out the sale of high wattage incandescent light bulbs. As these bulbs fail and supplies dwindle, consumers will eventually have to replace their high wattage incandescents with either lower wattage incandescent bulbs, compact fluorescent bulbs (CFLs), or Light Emitting Diodes (LEDs). All have disadvantages: lower wattage bulbs are dimmer; CFLs produce harsh, unnatural light, and take time to reach full intensity after they are powered up (especially in the cold); LEDs are substantially more expensive than incandescent bulbs, though their price is falling rapidly, and they last for many years.

Although this policy is designed to reduce aggregate electricity consumption and pollution, the policy will leave some consumers worse off (despite their slightly lower utility bills). How much would consumers need to be compensated to make them indifferent to the policy change? This calculation depends on the expenditure function. Let's say that consumer utility prior to the ban is given by $\bar{U}$ and expenditures by:

$$E_{pre} = E\left(p_{IN}, p_{CFL}, p_{LED}, p_a, \bar{U}\right),$$

where $p_{IN}$ is the price of an incandescent lightbulb, $p_{CFL}$ and $p_{LED}$ are the prices of CLFs and LEDs respectively, and $p_a$ is the price of all other goods. Notice that we do *not* assume that these three types of bulbs differ only according to price (i.e., that they are three 'quantities' of the same good). Because these lights have different characteristics, consumers will have different preferences over them.

To attain the same level of utility after the ban on high wattage incandescents, consumers would need this much income to be indifferent:

$$E_{post} = E\left(p_{IN} = \infty, p_{CFL}, p_{LED}, p_a, \bar{U}\right).$$

The difference $E_{post} - E_{pre}$ is the amount of money that we would need to compensate homeowners to leave them indifferent between the world with and without high wattage incandescent bulbs.

Of course, we don't usually know the expenditure function, so this isn't as easy to apply in practice as it is in theory. But it turns out that if we have an estimate of the compensated elasticity of demand for a good, this is often enough to make a rough calculation. We will discuss this in class.

One very crude estimate we could make at the outset is this. Let $x^*_{pre} = (x^*_{IN}, x^*_{CFL}, x^*_{LED}, x^*_a)$ equal the consumer's original *chosen* bundle (because this bundle is chosen, we are justified in treating it as their optimal choice), with $E_{pre} = x^*_{IN} \times p_{IN} + x^*_{CFL} \times p_{CFL} + x^*_{LED} \times p_{LED} + x^*_a \times p_a$. In the post period, we have $p'_{IN} = \infty$, meaning incandescent bulbs are no longer available. So we could calculate how much more income the consumer would need to substitute from incandescent to the cheapest alternative light source:

$$\Delta E' = E'_{post} - E \simeq x^*_{IN} \times \min\{p_{CFL}, p_{LED}\} + x^*_{CFL} \times p_{CFL} + x^*_{LED} \times p_{LED} + x^*_a \times p_a - E_{pre},$$

where I've assumed for simplicity that prices other than $p_{IN}$ are unchanged. Assuming that $\min\{p_{CFL}, p_{LED}\} > P_{IN}$, which implies that $\Delta E' > 0$. In this case, $\Delta E'$ could either underestimate or overestimate the amount of compensation needed to leave the consumer indifferent. It would tend to *underestimate* necessary compensation if the consumer prefers incandescent to CFL or LED light. In that case, simply providing sufficient resources to purchase CFL or LED bulbs in place of incandescent bulbs *might* leave the consumer worse off than she was initially. On the other hand, since $\Delta E' > 0$, the new budget would allow the consumer purchase more of $x_a$ (all other goods) than under the original budget set. It's possible that the consumer would then prefer a bundle that included more $x_a$ and fewer bulbs than in the original bundle, $x^*_{pre}$, and wouldn't be much worse off after the change. Thus, $\Delta E'$ is a crude estimate because we do not know if over or underestimates the total change.

In certain cases, we *know* that the transfer $\Delta E'$ underestimates the utility loss caused by the policy change. Take a case where, prior to the incandescent ban, $\{p_{CFL}, p_{LED}\} < P_{IN}$. That is, incandescent bulbs cost more than unconventional bulbs. This is plausible because CFLs and LEDs are heavily subsidized under some green energy programs; one can occasionally get CFLs or LEDs for free from the local power provider—and I'll pay you to take some of my old CFLs because I deeply dislike them but my wife won't let me throw them away. As long as the consumer was originally buying some positive number of incandescent bulbs, $x^*_{IN} > 0$ (that is, even with $p_{IN} > p_{CFL} = 0$, she still chose to buy at least some incandescent bulbs) our crude calculation above would imply that $\Delta E' < 0$, meaning that the consumer now needs *less* income to achieve the same utility. Mechanically, she will be buying cheaper CFLs rather than incandescent bulbs from now on, so she needs less money to purchase the same number of bulbs. By non-satiation, $\Delta E'$ strictly *underestimates* the compensation required to leave the consumer indifferent; the consumer is always worse off when the budget set is reduced since all *goods* have positive marginal utility. Another way to see this intuitively: the new hypothetical budget set, $E'_{post}$ excludes the originally chosen bundle $x^*_{pre}$ without allowing the consumer to purchase any previously unavailable bundle—that is, the new budget set is a strict subset of the old budget set. Since the original bundle was the consumer's chosen point in the original budget set, it must at least weakly dominate all other points in that set. Excluding that bundle from the budget set without introducing new choices must therefore make the consumer at least weakly worse off.[2]

[2] Technically, I'm using a different form of argument here called Revealed Preference, which was developed by Nobel prize winner Paul Samuelson as a weaker alternative (i.e., requiring fewer and weaker assumptions) to the axiomatic utility theory that we are using in class. I will not devote time to the theory of Revealed Preference this semester, but the idea is intuitive and can serve as a useful shortcut.

## 2 The Carte Blanche Principle

- One immediate implication of consumer theory is that consumers make optimal choices for themselves given prices, constraints, and income. [Generally, the only binding constraint is that they can't spend more their income, but we'll see examples where there are additional constraints.]

- This observation gives rise to the Carte Blanche principle: consumers are always weakly better off receiving a cash transfer than an in-kind transfer of identical monetary value. [Weakly better off in that they may be indifferent between the two.]

- With cash, consumers have Carte Blanche to purchase whatever bundle or goods are services they can afford – including the good or service that alternatively could have been transferred to them in-kind.

- Prominent examples of in-kind transfers given to U.S. citizens include Food Stamps, housing vouchers, health insurance (Medicaid), subsidized educational loans, child care services, job training, etc. [An exhaustive list would be long indeed.]

- Economic theory suggests that, relative to the equivalent cash transfer, these in-kind transfers serve as *constraints* on consumer choice. If consumers are rational, constraints on choice cannot be beneficial.

- For example, consider a consumer who has income $I = 100$ and faces the choice of two goods, food and housing, at prices $p_f, p_h$, each priced at 1 per unit. The consumer's problem is

$$\begin{aligned} & \max_{f,h} U(f,h) \\ s.t.\ \ f + h & \leq 100 \end{aligned}$$

- The government decides to provide a housing subsidy of 50. This means that the consumer can now purchase up to 150 units of housing but no more than 100 units of food. The consumer's problem is:

$$\begin{aligned} & \max_{f,h} U(f,h) \\ s.t.\ \ f + h & \leq 150 \\ f & \leq 50. \end{aligned}$$

- Alternatively, if the government had provided 50 dollars in cash instead, the problem would be:

$$\begin{aligned} & \max_{f,h} U(f,h) \\ s.t.\ \ f + h & \leq 150. \end{aligned}$$

- The government's transfer therefore has two components:

  1. An expansion of the budget set from $I$ to $I' = I + 50$.
  2. The imposition of the constraint that $f \leq 50$.

- The canonical economist's question is: why impose both (1) and (2) when you can just impose (1) and potentially improve consumer welfare at no additional cost to the government? (Of course, I don't expect you to accept this argument as gospel truth. But it's a good default position—better, perhaps, than the alternative default that it's better for the government to dictate choices to consumers than to allow them to make them for themselves.)

## 3 A Simple Example: The Deadweight Loss of Christmas

- Joel Waldfogel's 1993 *American Economic Review* paper provides a stylized (and surprisingly controversial) example of the application of the Carte Blanche principle.

- Waldfogel observes that gift-giving is equivalent to an in-kind transfer and hence should be less efficient for consumer welfare than simply giving cash.

- In January, 1993, he asked the following questions of approximately 150 Yale undergraduates about holiday gifts received in 1992:

  1. What were the gifts worth in cash value (purchase price)?
  2. How much would the students be willing to pay for these gifts if they didn't already have them?
  3. How much would the students we willing to accept in cash in lieu of the gifts? (Usually higher than willingness to pay – an economic anomaly.)

- For each gift, Waldfogel calculated the gift's "yield," $Y_j = V_j/P_j$, where $P_j$ is the purchase price and $V_j$ is the student's willingness to pay.

- As theory (and intuition) would predict, the yield was, on average, well below one-hundred percent. Waldfogel concludes that in-kind gift giving "destroys" economic value relative to the cost-equivalent cash gift.

- Figure I of Waldfogel illustrates the idea transparently:

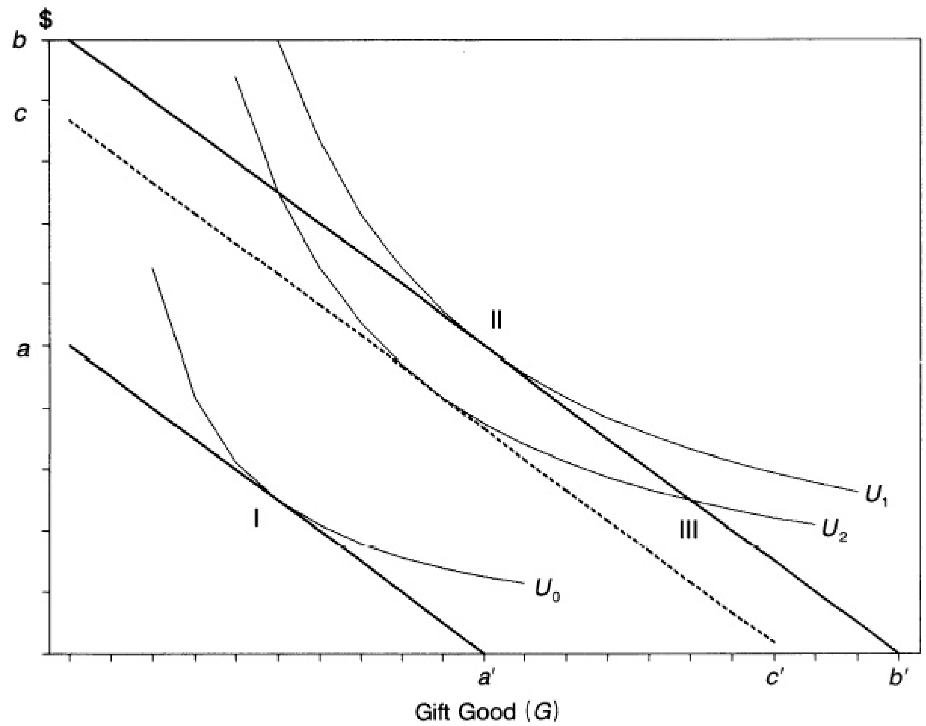



FIGURE 1. GIFT-GIVING AND DEADWEIGHT LOSS

- The budget line $aa'$ is the original budget set.
- The line $bb'$ is the budget set for an in-kind transfer.
- $U_1$ is the highest feasible indifference curve achievable for this consumer with budget set $bb'$. This is achieved with consumption bundle $II$.
- The intersection of $U_2$ and $bb'$, labeled $III$, is the consumption bundle with the in-kind gift. The amount of $G$ is selected by the gift-giver rather than the recipient.
- Although $III$ lies on $bb'$, it is not on the highest achievable indifference curve achievable with budget set $bb'$.
- Line $cc'$ is the actual budget set the consumer would require to attain utility $U_2$ if his choice set were not constrained by the gift giver.
- The "deadweight loss" of gift-giving relative to the equivalent cash transfer in this example is equal to $(b' - c') \times p_g$.

**Several interesting observations from the article:**

1. Value "destruction" is greater for distant relatives, e.g., grandparents.
2. Value "preservation" is near-perfect for friends
3. Groups that tend to "destroy" the most value are the most likely to give cash instead

**It's useful to interpret the basic regression result given on the top of page 1332:**

$$\begin{array}{ccccc} \ln(value_i) & = & -0.314 & + & 0.964 \ln(price_i) \\ & & (0.44) & & (0.08) \end{array} .$$

- The quantities in parentheses are standard errors. Since 0.964 is much larger than $2 \times 0.08$, the relationship between value and price is highly statistically significant.
- The derivative of value with respect to price is (recall that $\partial / \partial x$ of $\ln x$ is $\partial x / x$):

$$\frac{\partial \ln(value_i)}{\partial \ln(price_i)} = \frac{\partial value_i}{value_i} \cdot \frac{price_i}{\partial price_i} = 0.964.$$

  That is, a 1 percent rise in price translates into a 0.964 percent rise in value.

- But, there is a major difference between the value and price. Rewriting the equation and exponentiating:

$$\begin{aligned} \ln(\text{value}_i) &= \ln(\exp(-0.314)) + 0.964 \ln(\text{price}_i) \\ &= \ln\left(\exp(-0.314) \times \text{price}_i^{0.964}\right) \end{aligned}$$

$$\begin{aligned} \text{value}_i &= \exp(-0.314) \times \text{price}_i^{0.964} \\ &= 0.73 \times \text{price}_i^{0.964} \end{aligned}$$

- So, for a \$100 gift, the approximate recipient valuation is about \$62.
- You can see why it's handy to use natural logarithms to express these relationships. They readily allow for proportional effects. The regression equation above says that the value of a gift is *approximately* equal to 96% of its price minus approximately 31 percent (logarithmic transformations are non-linear, which is why we emphasize the word approximate).
- The Waldfogel article generated a surprising amount of controversy, even among economists, most of whom probably subscribe to the Carte Blanche principle. But to many non-economist readers, this article seems to exemplify the well-worn gripe about economists, "They know the price of everything and the value of nothing."
- *What is Waldfogel missing?*