

14.03/14.003 Problem Set 2 SOLUTIONS

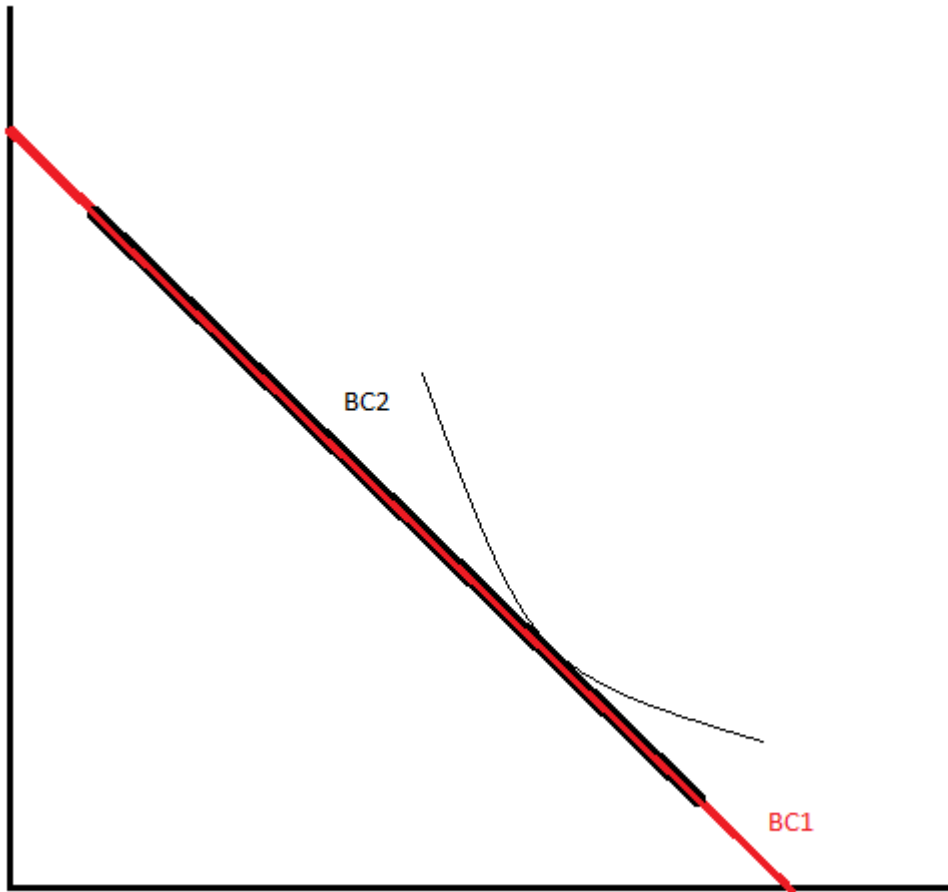
Fall 2014

Professor David Autor

1 Short Answer Questions

1. Alice only consumes Sushi (S) and Kind Bars (K), and spends all of her income on the two goods. Under what conditions would a government gift of 2 S and 2 K (which cost a total of \$10 on the market) be as good in terms of Alice's well being as a gift of \$10?

[Solution (4 points)]: Alice is indifferent to the gifts if at income level $I+10$, her optimal choice of S and K are each at least 2. In the following graph, BC1 represents Alice's budget constraint with a gift of \$10 and BC2 is her budget constraint with a gift of 2S and 2K. If her optimal consumption on BC1 is also on BC2, then both gifts are equally valuable to Alice.



2. Bob also only consumes only Sushi and Kind Bars.

- (a) True, false, or indeterminate **and** why: If the price of S doubles, Bob's marginal rate of substitution at his original optimal level of consumption changes.

[Solution (1 point)]: False. Bob's marginal rate of substitution is not a function of prices, it only depends on his preferences.

- (b) What is the MRS of S at the new optimal bundle compared to the old one?

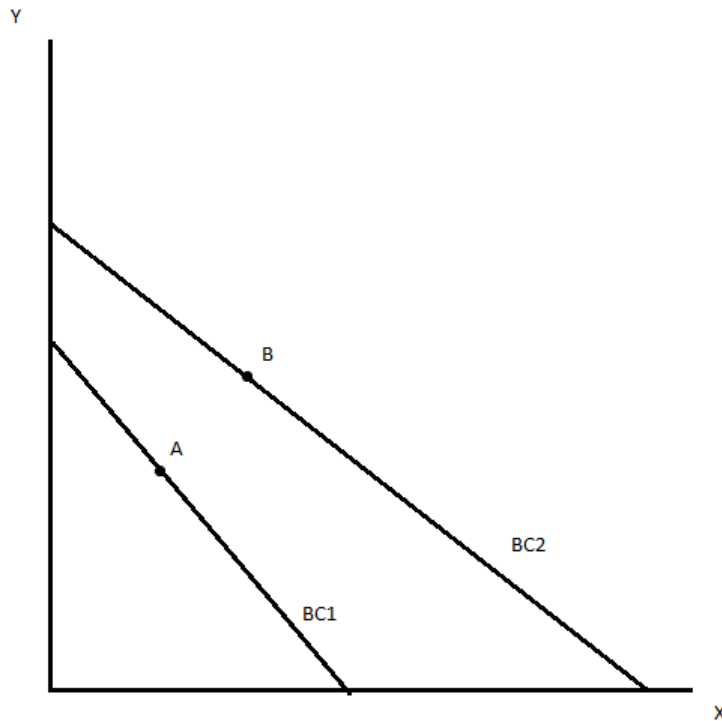
(1 point) We know that the MRS at the optimal point is equal to the ratio of prices: $\frac{MU_S}{MU_K} = \frac{P_S}{P_K}$. When P_S doubles, MRS at the new optimal must also double, so we can conclude that the MRS is twice as high in the new optimal bundle.

- (c) In this scenario, you find that Bob's consumption of K decreased. What are the substitution and income effects on K that would explain this? What do we call this kind of good?

(2 points) When the price of S increased, the substitution effect increases our consumption of K. If we find that the consumption of K decreased, we know

that the income effect must be driving the decrease. Our income decreased, causing our consumption of K to decrease, which is indicative of a normal good.

3. In the following graph, we have two price ratios and budget constraints, labeled BC1 and BC2. The optimal levels of consumption chosen on BC1 and BC2 are A and B, respectively. Describe the change in income I and the price ratio p_x/p_y that would move the budget from BC1 to BC2. Can we determine if Y is a normal or inferior good?

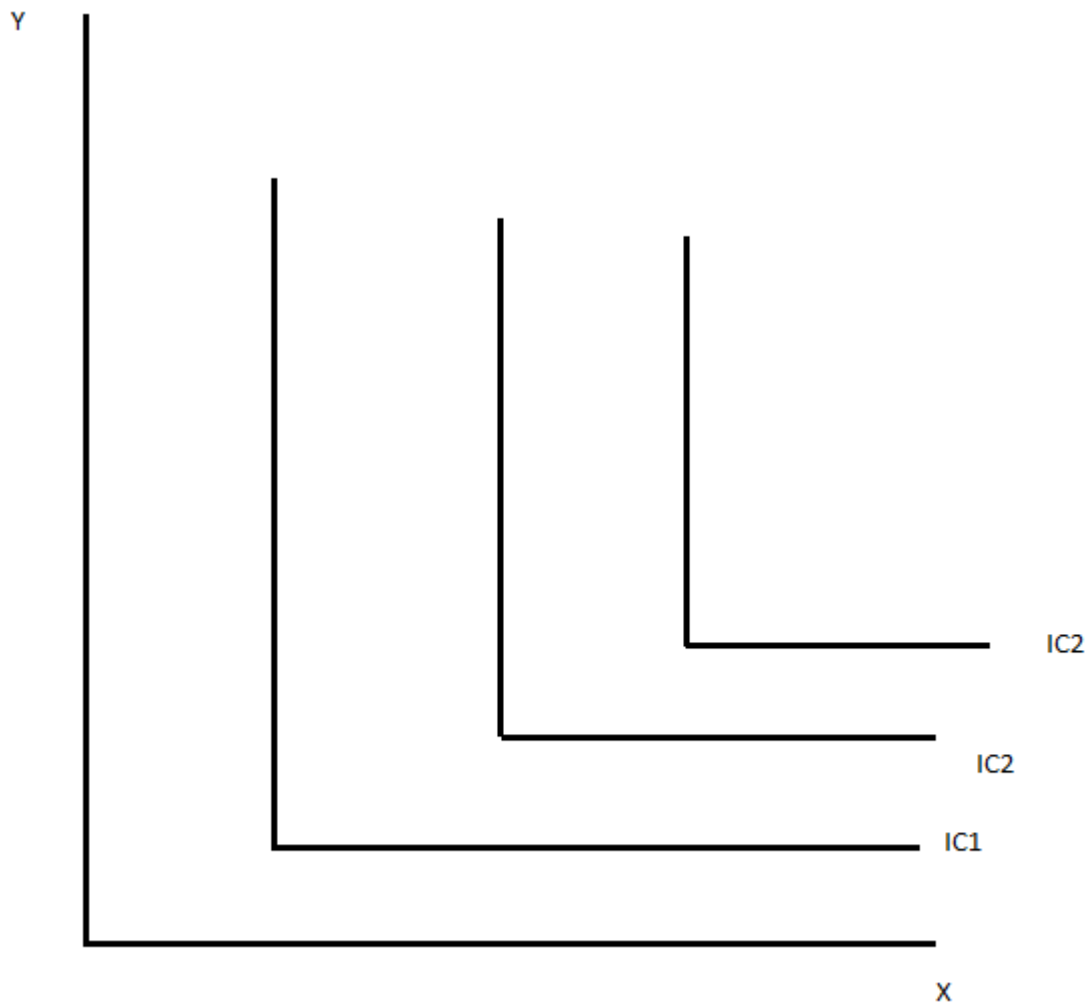


[Solution (4 points)]: there are a number of ways we could move from BC1 to BC2. I could increase, and p_x decrease. Or if I stays constant and p_x and p_y both decrease and p_x/p_y decreases.

Y is a normal good. Moving from BC1 to BC2, p_x/p_y decreases so the substitution effect causes less consumption of Y. Because we observe more consumption of Y, the income effect must be dominant. The rise in income caused consumption of Y to increase, so Y is a normal good.

4. Carol has utility function $u(X,Y)=\min(X,2Y)$. Draw indifference curves for this utility. Which of the 5 axioms of consumer preferences (if any) does Carol violate?

[Solution (4 points)]:

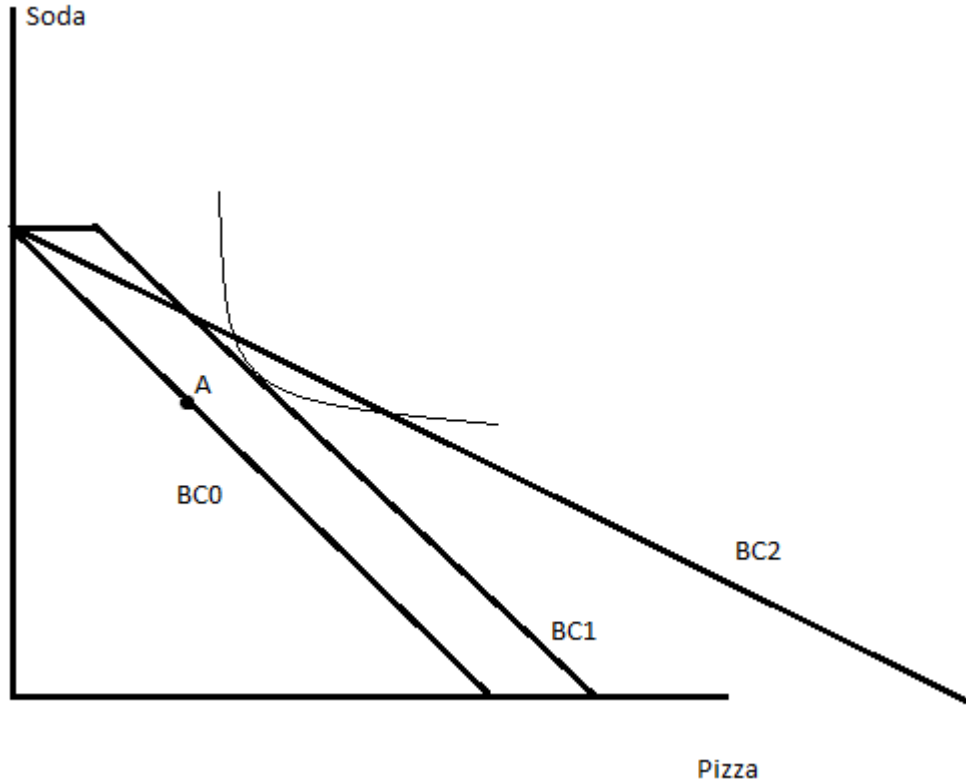


Axiom 4, Non Satiation. Carol is indifferent between the points (6,2) and (8,2), which is a violation of non-satiation.

5. In response to widespread student malnutrition at MIT, President Reif implements a pizza subsidy program that reduces the price of a slice of a pizza from \$2 to \$1. Prior to the subsidy program, Carlos (an MIT student with standard consumer preferences that satisfy all five axioms) bought 2 slices of pizza every day and spent the rest of his money on Soda. Pizza is a normal good for Carlos. True, false, indeterminate and **why**: Carlos is indifferent between the subsidy program and a program that gives him 1 free slice of pizza each day. (Assume that Carlos can buy pizza in any increment, e.g., he can buy 10% of a slice for 10% of the cost of a whole slice).

[Solution (4 points)]: False. In the following figure, BC0 is Carlos' initial budget constraint, with consumption of 2 slices of pizza at point A. BC1 is his budget constraint with a free slice of pizza, and BC2 is his budget constraint under the subsidy. Moving from BC0 to BC1 is an increase in income, and because pizza

is a normal good, Carlos should choose to purchase more pizza on BC1. If his indifference curves are tangent to BC1 at any point to the right of A, then those indifference curves will be inside the budget constraint BC2, and Carlos's welfare can be improved by moving to the subsidy BC2.



2 Income and Substitution Effects

A consumer has utility function

$$U(x, y) = \left(\frac{x}{2}\right)^{\frac{1}{4}} y^{\frac{3}{4}}$$

1. Suppose prices are $P_x = 3$, $P_y = 1$ and income $I = 30$. Set up the consumer's constrained maximization problem. Solve for the optimal consumption bundle by first taking a log-transformation of the utility function. Why is it okay to take logs of the utility function for solving the optimization problem?

[**Solution (4 points):** maximization problem is

$$\begin{aligned} \max_{x,y} & \left(\frac{x}{2}\right)^{\frac{1}{4}} y^{\frac{3}{4}} \\ \text{s.t.} & P_x x + P_y y = I \Rightarrow 3x + y = 30 \end{aligned}$$

We can take a log-transformation of utility and get the same solution since utility is ordinal, not cardinal, and this transformation is positive monotonic (increasing), so it preserves the

original preference ordering. Taking logs the problem becomes

$$\begin{aligned} \max_{x,y} & \frac{1}{4} \ln \left(\frac{x}{2} \right) + \frac{3}{4} \ln y \\ \text{s.t.} & P_x x + P_y y = I \end{aligned}$$

(We will plug in P_x , P_y and I later). The Lagrangian is

$$\mathcal{L} = \frac{1}{4} \ln \left(\frac{x}{2} \right) + \frac{3}{4} \ln y + \lambda (I - P_x x - P_y y)$$

FOCs are

$$\begin{aligned} \frac{1}{4x} &= P_x \lambda \\ \frac{3}{4y} &= P_y \lambda \\ P_x x + P_y y &= I \end{aligned}$$

This is a system of three equations in three unknowns (x, y, λ) so it can be solved by rearranging the equations. The most efficient way of rearrangement in utility maximization problems is to divide the first two first-order conditions, since that allows you to drop λ and solve a system with two unknowns first. This approach gives us

$$\frac{4y}{12x} = \frac{P_x}{P_y} \Rightarrow y = 3x \frac{P_x}{P_y}$$

Now using the budget constraint

$$P_x x + P_y 3x \frac{P_x}{P_y} = I \Rightarrow x^* = \frac{I}{4P_x} = \frac{30}{12} = 2.5$$

And again from the budget constraint we have that

$$y^* = \frac{I - P_x x}{P_y} = \frac{3I}{4P_y} = \frac{90}{4} = 22.5$$

Notice that the form of preferences are such that the consumer optimally spends $\frac{1}{4}$ of income on x and $\frac{3}{4}$ of income on y .]

- Find the Marshallian demand curves d_x and d_y for x and y given prices P_x , P_y and income I .
[**Solution (4 points):** we have this from the previous question:

$$\begin{aligned} d_x &= \frac{I}{4P_x} \\ d_y &= \frac{3I}{4P_y} \end{aligned}$$

]

3. Given the demand curve you found for x , determine the uncompensated price effect, that is $\frac{\delta d_x}{\delta P_x}$.

[**Solution (2 points):** Here we just take the derivative of demand with respect to the price, giving us: $\frac{\delta d_x}{\delta P_x} = -\frac{1}{4} \frac{I}{P_x^2}$ which is negative.]

4. Given the demand curve you found for x , determine the income effect, that is, $\frac{\delta d_x}{\delta I} \times \frac{\delta E}{\delta P_x}$ (where E is the expenditure function).

[**Solution (5 points):** To find $\frac{\delta E}{\delta P_x}$ (which by Shephard's lemma is equal to Hicksian demand), we need to either find and differentiate the expenditure function or find Hicksian demand from the expenditure minimization problem. We will show both approaches here. **1st approach:** to find the expenditure function we use the duality result, i.e. that

$$v(p, I) = u_0 \iff v(p, e(p, u_0)) = u_0$$

where

$$v(p, I) = \left(\frac{x^*}{2}\right)^{\frac{1}{4}} \left(y^{\frac{3}{4}}\right)^{\frac{1}{4}} = \left(\frac{I}{8P_x}\right)^{\frac{1}{4}} \left(\frac{3I}{4P_y}\right)^{\frac{3}{4}} = I \left(\frac{1}{8}\right)^{\frac{1}{4}} \left(\frac{3}{4}\right)^{\frac{3}{4}} \frac{1}{P_x^{\frac{1}{4}} P_y^{\frac{3}{4}}} = u_0$$

so it follows that

$$e(p, u_0) = u_0 P_x^{\frac{1}{4}} P_y^{\frac{3}{4}} 8^{\frac{1}{4}} \left(\frac{4}{3}\right)^{\frac{3}{4}} = E$$

and now differentiating to get Hicksian demand for x we have

$$\begin{aligned} \frac{\delta E}{\delta P_x} &= u_0 \frac{1}{4} \left(\frac{P_y}{P_x}\right)^{\frac{3}{4}} 8^{\frac{1}{4}} \left(\frac{4}{3}\right)^{\frac{3}{4}} \\ &= u_0 \left(\frac{P_y}{P_x}\right)^{\frac{3}{4}} \left(4^{\frac{1}{4}} 2^{\frac{1}{4}} 4^{\frac{3}{4}} \left(\frac{1}{3}\right)^{\frac{3}{4}} \frac{1}{4}\right) = u_0 \left(\frac{P_y}{P_x}\right)^{\frac{3}{4}} \left(2^{\frac{1}{4}} \left(\frac{1}{3}\right)^{\frac{3}{4}}\right) \end{aligned}$$

Thus, in the general form, the income effect of interest is:

$$\frac{\delta d_x}{\delta I} \frac{\delta E}{\delta P_x} = \frac{1}{4P_x} u_0 \left(\frac{P_y}{P_x}\right)^{\frac{3}{4}} \left(2^{\frac{1}{4}} \left(\frac{1}{3}\right)^{\frac{3}{4}}\right)$$

As you saw in lecture notes, the Hicksian demand is equal to Marshallian demand if evaluated at the equilibrium levels of utility/expenditure, so if you were interested in just finding the level of the income effect at a specific point rather than the whole functional form, substituting for the Marshallian demand evaluated at fixed income and prices would have been sufficient.

2nd approach: The consumer's expenditure minimization problem is:

$$\begin{aligned} &\min P_x x + P_y y \\ &\text{s.t. } \frac{1}{4} \ln \left(\frac{x}{2}\right) + \frac{3}{4} \ln y = u_0 \end{aligned}$$

The Lagrangian is then:

$$\mathcal{L} = P_x x + P_y y + \lambda \left(u_0 - \frac{1}{4} \ln \left(\frac{x}{2} \right) \left(-\frac{3}{4} \ln y \right) \right)$$

The FOCs are:

$$P_x = \frac{1}{4} \lambda \frac{1}{x}$$

$$P_y = \frac{3\lambda}{4y}$$

$$u_0 = \left(\frac{x}{2} \right)^{\frac{1}{4}} \left(y^{\frac{3}{4}} \right)$$

It follows that

$$\frac{P_x}{P_y} = \frac{y}{3x}$$

$$u_0 = \left(\frac{x}{2} \right)^{\frac{1}{4}} \left(y^{\frac{3}{4}} \right)$$

Thus, $y = \frac{3xP_x}{P_y}$. Substituting this into the constraint we get: $u_0 = \left(\frac{x}{2} \right)^{\frac{1}{4}} \left(\frac{3xP_x}{P_y} \right)^{\frac{3}{4}}$, and rearranging further we have

$$u_0 = \left(\frac{1}{2} \right)^{\frac{1}{4}} \left(\frac{3P_x}{P_y} \right)^{\frac{3}{4}} x$$

$$\Rightarrow u_0 (P_y)^{\frac{3}{4}} = \left(\frac{1}{2} \right)^{\frac{1}{4}} 3^{\frac{3}{4}} (P_x)^{\frac{3}{4}} x$$

$$\Rightarrow h_x = u_0 2^{\frac{1}{4}} \left(\frac{1}{3} \right)^{\frac{3}{4}} \left(\frac{P_y}{P_x} \right)^{\frac{3}{4}}$$

which is the Hicksian demand for x (and the same as we got from the first approach). Thus we again get the income effect as:

$$\frac{\delta d_x}{\delta I} \frac{\delta E}{\delta P_x} = \frac{1}{4P_x} \frac{u_0}{\left(\frac{1}{2} \right)^{\frac{1}{4}} 3^{\frac{3}{4}}} \left(\frac{P_y}{P_x} \right)^{\frac{3}{4}}$$

]

5. Using the Slutsky equation, find the substitution effect, $\frac{\delta d_x}{\delta P_x} |_{U=const}$.

[**Solution (2 points):** using the Slutsky equation we have

$$\frac{\delta d_x}{\delta P_x} |_{U=const} = \frac{\delta d_x}{\delta P_x} + \frac{\delta d_x}{\delta I} d_x = -\frac{I}{4P_x^2} + \frac{I}{(4P_x)^2} = -\frac{3I}{16P_x^2}$$

Note that the substitution effect is always negative.]

6. In general, is x a normal good? Why or why not?

[**Solution (3 points):** remember, the income effect must be strictly positive for a normal good and strictly negative for an inferior good. Here the income effect is always positive and therefore x is a normal good.]

3 Indirect Utility and Expenditure Functions

In an economy with two goods, fish (f) and chips¹ (c), a consumer has the following indirect utility function

$$v(p_c, p_f, I) = \ln(I) - \ln p_c - (1 - \alpha) \ln p_f \quad (1)$$

where p_f and p_c are the prices of fish and chips, I denotes income, and $\alpha \in (0, 1)$ is a parameter.

1. Recalling that the indirect utility function comes from

$$v(p_c, p_f, I) = \max_{c, f} u(c, f) \quad (2)$$

$$\text{s.t. } I = p_c c + p_f f$$

use the Envelope Theorem to obtain $\frac{\delta v}{\delta p_c}$, $\frac{\delta v}{\delta p_f}$ and $\frac{\delta v}{\delta I}$ in the general problem (2) and in our particular example (1) [Hint: set up the Lagrangian]. Use the results to obtain the Marshallian demands of this consumer.

[**Solution (5 points):** For constrained optimization problems, the Envelope Theorem allows us to obtain derivatives of a value function (such as $v(p, I)$ in the problem) by taking derivatives of the Lagrangian, treating the solution as fixed. That means, we do not need to consider effects from $\frac{\delta c^*}{\delta p_c}$, $\frac{\delta f^*}{\delta I}$ and other similar terms as they do not have any effect on first derivatives.

For the general problem the Lagrangian is

$$\mathcal{L} = u(c, f) + \lambda [I - p_c c - p_f f]$$

Applying the Envelope Theorem we have

$$\frac{\delta v(p_c, p_f, I)}{\delta p_c} = \frac{\delta \mathcal{L}}{\delta p_c} = -\lambda^*(p, I) c^*(p, I)$$

$$\frac{\delta v(p_c, p_f, I)}{\delta p_f} = \frac{\delta \mathcal{L}}{\delta p_f} = -\lambda^*(p, I) f^*(p, I)$$

$$\frac{\delta v(p_c, p_f, I)}{\delta I} = \frac{\delta \mathcal{L}}{\delta I} = \lambda^*(p, I)$$

¹In the British sense, not that this matters.

We can use these results (or Roy's Identity) to obtain

$$c^*(p, I) = -\frac{\frac{\delta v}{\delta p_c}}{\frac{\delta v}{\delta I}}, \quad f^*(p, I) = -\frac{\frac{\delta v}{\delta p_f}}{\frac{\delta v}{\delta I}}$$

In our particular example then we have

$$\begin{aligned} \frac{\delta v(p_c, p_f, I)}{\delta p_c} &= -\frac{1}{p_c} \\ \frac{\delta v(p_c, p_f, I)}{\delta p_f} &= -\frac{(1-\alpha)}{p_f} \\ \frac{\delta v(p_c, p_f, I)}{\delta I} &= \frac{1}{I} \end{aligned}$$

therefore the Marshallian demands are

$$c^*(p, I) = \frac{I}{p_c}, \quad f^*(p, I) = \frac{(1-\alpha)I}{p_f}$$

]

2. Find the expenditure function.

[Solution (5 points): We use the duality results, in particular,

$$v(p, I) = \underline{u} \iff v(p, e(p, \underline{u})) = \underline{u}$$

to obtain

$$\underline{u} = \ln e(p, \underline{u}) = \ln p_c - (1-\alpha) \ln p_f$$

Then, we simply need to invert that expression. We apply the exponential (after using log rules to note that $\underline{u} = \ln \frac{e(p, \underline{u})}{p_c p_f^{1-\alpha}}$) and get

$$\exp(\underline{u}) = \frac{e(p, \underline{u})}{p_c p_f^{1-\alpha}}$$

$$\Rightarrow e(p, \underline{u}) = \exp(\underline{u}) p_c p_f^{1-\alpha}$$

which is the expenditure function.]

3. With another application of the Envelope Theorem, derive the Hicksian demands from the expenditure function you have just obtained.

[Solution (5 points): The Hicksian Demands can be obtained by using Shephard's Lemma, which indicates that $h_f(p, \underline{u}) = \frac{\delta e}{\delta p_f}$ and $h_c(p, \underline{u}) = \frac{\delta e}{\delta p_c}$. That can be verified by studying the Lagrangian associated to the expenditure minimization problem, which is of the form

$\mathcal{L} = p_f f + p_c c + \mu [\underline{u} - u(x, y)]$. In this particular case, the Hicksian demands are given by

$$h_c(p, \underline{u}) = \frac{\delta e}{\delta p_c} = \exp(\underline{u}) \left(\frac{p_f}{p_c} \right)^{1-}$$

$$h_f(p, \underline{u}) = \frac{\delta e}{\delta p_f} = (1 -) \exp(\underline{u}) \left(\frac{p_c}{p_f} \right) \left($$

]

4. Suppose you learn that this consumer initially spends $\frac{1}{3}$ of her income on fish and $\frac{2}{3}$ of her income on chips. What would be your best estimate of in equation (1)? Use this value of for the next subquestion.

[Solution (5 points): using the Marshallian demand from (1) we have that

$$c = \frac{I}{p_c} \Rightarrow = \frac{cp_c}{I} = \frac{2}{3}$$

In this example, the consumer always spends proportion of her income on chips and remaining proportion $(1 -)$ on fish. So our best estimate is $= \frac{2}{3}$.]

5. Assume that initial prices are $p_f = p_c = 1$, initial income is $I = 900$ and the price of fish increases to $p'_f = 1.75$ after a toxic waste spill kills some of the fish. How much additional income would be necessary to bring this consumer back to the welfare level achieved under the initial prices?

[Solution (5 points): The initial level of utility achieved is

$$v(1, 1, 900) = \ln 900 - \ln 1 - (1 -) \ln 1 = \ln 900$$

and we want to find I^{new} such that

$$v(1, 1.75, I^{new}) = v(1, 1, 900)$$

So I^{new} solves

$$\ln I^{new} - \frac{2}{3} \ln 1 - \frac{1}{3} \ln 1.75 = \ln 900$$

$$\Rightarrow \ln I^{new} = \ln 900 + \frac{1}{3} \ln 1.75 \approx 6.99$$

It follows that $I^{new} \approx \$1086$. The additional income necessary to bring the consumer back to the initial welfare level is $\$1086 - \$900 = \$186$.]

4 Cash for Clunkers

4.1 Theory

Matt and David are two consumers who love cars (C) and food (F) (and nothing else!). Both of them live for two periods, period 1 and period 2. They have same utility functions $U(C_t, F_t)$ in each period and have same income (I) each period. They both face same prices of cars and food: p_{F1} and p_{F2} for food and p_{C1} and p_{C2} for cars. The subscripts 1 and 2 indicate the period. Both Matt and David know all the prices in period 0, before period 1 begins.

But they are very different in how they go about living their lives. Matt spends his entire income each period. David, on the other hand, decides in period 1 how he will spend his total income ($2I$) across both periods. (Assume that he can borrow or save as needed between periods at no cost).

1. **(1 point)** Write down Matt's optimization problem in period 1.

Matt's problem is:

$$\max U(C_1, F_1) \text{ such that } p_{C1}C_1 + p_{F1}F_1 = I$$

The associated Lagrangian is

$$\mathcal{L}(C_1, F_1, \lambda) = U(C_1, F_1) + \lambda(I - p_{C1}C_1 - p_{F1}F_1)$$

2. **(1 point)** Write down Matt's optimization problem in period 2. You need to set up the Lagrangian but do not need to solve.

The optimization problem in period 2 is identical except for the change in prices.

$$\max U(C_2, F_2) \text{ such that } p_{C2}C_2 + p_{F2}F_2 = I$$

The associated Lagrangian is

$$\mathcal{L}(C_2, F_2, \lambda) = U(C_2, F_2) + \lambda(I - p_{C2}C_2 - p_{F2}F_2)$$

3. **(1 point)** Explain whether his demand for car in period 2 depends upon the price of cars in period 1.

The Lagrangian in period 2 optimization does not depend upon period 1 prices. Therefore, Matt's demand function will only depend on the income I and period 2 prices of food and cars. Hence, Matt will not react to changes in period 1 prices by altering his consumption in period 2.

4. **(2 points)** Assuming that David wants to maximize the sum total of utilities in two periods, write down his optimization problem and the associated Lagrangian.

David's problem is:

$$\begin{aligned} \max U(C_1, F_1) + U(C_2, F_2) \\ \text{such that } p_{C1}C_1 + p_{F1}F_1 + p_{C2}C_2 + p_{F2}F_2 = 2I \end{aligned}$$

and the associated Lagrangian is

$$\mathcal{L}(C_1, F_1, C_2, F_2, \lambda) = U(C_1, F_1) + U(C_2, F_2) + \lambda(2I - (p_{C1}C_1 + p_{F1}F_1 + p_{C2}C_2 + p_{F2}F_2))$$

5. **(1 points)** True/false/uncertain **and** why: David's demand for car in period 2 does not depend upon the price of cars in period 1.

False. Generically, the demand functions from the optimization problem above will depend upon all the prices and income. So, when period 1 price of car changes, it will affect David's demand for cars in period 2. (Plug in any utility function here, say $U(C_t, F_t) = C_t \cdot F_t$, to convince yourself).

Since David cares about the total utility across two periods, he will shift income and consumption optimally across the periods to maximize his total utility. Any changes in price of one good will affect the demand of other good in the same period (income and substitution effect) as well as across periods (inter-temporal substitution effect).

However, it is possible to construct utility functions in which the above statement does not hold. For instance, if David got zero utility from consumption (of cars) in period 2 ($U(C_2, F_2) = 0$) then, he would always consume everything in period 1.

6. **(1 point)** Further assume that cars bought in period 1 gives additional utility in period 2 (because you still haven't scrapped the car). How will David allocate income between two periods in light of this fact?

He will shift the purchase of cars to period 1 more because cars bought in period 1 will provide utility in two periods whereas cars bought in period 2 will provide utility for just one period.

7. **(2 points)** How will it affect David's demand for cars (in both periods) when the price of car goes down in period 1?

*When price of car goes down in period 1, he will demand more cars in period 1 (substitution effect). The fall in price of cars in period 1 makes David richer (income effect) in period 1. He will optimally shift some of the increased income to period 2 as well, increasing the demand for cars in period 2. **(1 point)***

But since cars bought in period 1 also provide utility in period 2, he will also want to shift more income in period 1. On the net, it could be possible that it is optimal to shift quite a lot of income in period 1 so that period 2 income decreases (from what was allocated before the

price change). Then it would lead to a fall in demand in cars in period 2. This is what Sufi and Mian find in their study. **(1 point)**

4.2 Empirics

Read the paper “The effects of fiscal stimulus: Evidence from the 2009 cash for clunkers program” by Atif Mian and Amir Sufi posted on the course website. The paper evaluates the impact of the 2009 Cash for Clunkers program that subsidized purchase of energy-efficient autos. Let $t = 0$ represent the period in which the policy is announced, $t = 1$ is the period in which the policy is in being implemented, and $t = 2$ as the period in which the policy stopped being implemented. Let $Y_{i,t}$ represents total car purchases in city i at time t .

1. Suppose you were in control of how the policy was implemented, and your goal was to evaluate the causal effect of the program on car consumption under credible assumptions. You choose to implement the subsidy in half of the US cities selected randomly, where the other half receives no subsidy.

- (a) **(2 points)** Let $X_i \in \{0, 1\}$ be an indicator of whether city i had the policy implemented, what comparison would you make to find the effect of the policy in period 1? What comparison would you make to find the effect of the policy in period 2?

The effect in period 1 is simply the difference in outcomes (sales of fuel efficient autos) between the two groups in period 1. That is

$$E[Y_{i,t=1}|X_i = 1] - E[Y_{i,t=1}|X_i = 0]$$

(1 point)

For period 2 effect, we would simply compare outcomes in period 2. That is:

$$E[Y_{i,t=2}|X_i = 1] - E[Y_{i,t=2}|X_i = 0]$$

(1 point)

Another correct estimate is doing the Diff-in-Diff estimates for each period instead of just the mean comparisons as above.

- (b) **(3 points)** Show mathematically, along with clearly stated assumptions, that the comparison in (a) would be the causal effect of the policy

Using Rubin’s Causal Framework, the difference for period 1 is

$$\begin{aligned} & E[Y_{i,t=1}|X_i = 1] - E[Y_{i,t=1}|X_i = 0] \\ &= E[Y_{1i,t=1}|X_i = 1] - E[Y_{0i,t=1}|X_i = 0] \\ &= E[Y_{1i,t=1}|X_i = 1] - E[Y_{0i,t=1}|X_i = 1] \\ &\quad + E[Y_{0i,t=1}|X_i = 1] - E[Y_{0i,t=1}|X_i = 0] \end{aligned}$$

Note that the first two terms

$$E[Y_{1i,t=1}|X_i = 1] - E[Y_{0i,t=1}|X_i = 1]$$

is the Average Treatment effect on the Treated, a causal effect and the last pair of terms is the selection bias:

$$+ E[Y_{0i,t=1}|X_i = 1] - E[Y_{0i,t=1}|X_i = 0]$$

which is the difference between the treatment and control groups had there been no treatment. (**2 points**)

Randomization ensures that all characteristics are the same in the treatment and control groups in period 0 (when randomization was done). Therefore, it is very likely that what happened in the control group (the second term) is, on average, the same as what would have happened to the treatment group if they had not been treated (the first term). (**1 point**)

Therefore, the selection bias is zero, and the comparison gives the causal estimate.

2. Unfortunately, that is not how the program was rolled out. In a simplified sense, the policy was implemented ($X_i = 1$) in cities with ‘high’ number of pre-existing clunkers (as of period 0) but was not implemented ($X_i = 0$) in cities with ‘low’ number of pre-existing clunkers (as of period 0). Assume that all cities in the US can be categorized into one of the two categories.

- (a) (**2 points**) Suppose you estimated

$$E[Y_{i,t=1}|X_i = 1] - E[Y_{i,t=1}|X_i = 0]$$

using the data on auto sales in all the cities. Show mathematically, what assumptions are needed for this estimate to be a causal estimate of the policy in period 1.

Using the RCM framework, the comparison is:

$$\begin{aligned} & E[Y_{i,t=1}|X_i = 1] - E[Y_{i,t=1}|X_i = 0] \\ &= E[Y_{1i,t=1}|X_i = 1] - E[Y_{0i,t=1}|X_i = 0] \\ &= E[Y_{1i,t=1}|X_i = 1] - E[Y_{0i,t=1}|X_i = 1] \\ &\quad + E[Y_{0i,t=1}|X_i = 1] - E[Y_{0i,t=1}|X_i = 0] \end{aligned}$$

As before, the selection bias is

$$+ E[Y_{0i,t=1}|X_i = 1] - E[Y_{0i,t=1}|X_i = 0]$$

(**1 point**)

We need to assume that selection bias is zero for the estimate to be valid. In this context, this means that auto sales in cities that have high number of clunkers in period 0 would have been the same as the **level** of auto sales in cities that have low number of clunkers in period 0. (1 point)

- (b) (1 point) Is the assumption in (a) likely to be satisfied? Answer yes, no, or indeterminate and explain why.

This assumption is unlikely to be satisfied, as cities with high amount of clunkers in period 0 will continue to have high level of clunkers in period 1 and 2 as well.

- (c) (3 points) The authors estimate a variant of the following estimate

$$E[Y_{i,t=1} - Y_{i,t=0}|X_i = 1] - E[Y_{i,t=1} - Y_{i,t=0}|X_i = 0]$$

Show mathematically, the assumptions needed for this estimate to be a causal estimate of the policy in period 1. Also, state the assumption that you are making verbally. Is it likely to be satisfied?

The estimate is

$$\begin{aligned} & E[Y_{i,t=1} - Y_{i,t=0}|X_i = 1] - E[Y_{i,t=1} - Y_{i,t=0}|X_i = 0] \\ = & E[Y_{1i,t=1} - Y_{0i,t=0}|X_i = 1] - E[Y_{0i,t=1} - Y_{0i,t=0}|X_i = 0] \\ = & E[Y_{1i,t=1} - Y_{0i,t=1} + Y_{0i,t=1} - Y_{0i,t=0}|X_i = 1] - E[Y_{0i,t=1} - Y_{0i,t=0}|X_i = 0] \\ = & E[Y_{1i,t=1} - Y_{0i,t=1}|X_i = 1] \\ & + E[Y_{0i,t=1} - Y_{0i,t=0}|X_i = 1] - E[Y_{0i,t=1} - Y_{0i,t=0}|X_i = 0] \end{aligned}$$

Here, the first term $E[Y_{1i,t=1} - Y_{0i,t=1}|X_i = 1]$ is the causal effect of the policy and the bias is

$$E[Y_{0i,t=1} - Y_{0i,t=0}|X_i = 1] - E[Y_{0i,t=1} - Y_{0i,t=0}|X_i = 0]$$

We need the bias term to be zero for the estimate to be a causal estimate in period 1. (1 point)

*This means that the observed change in outcomes between period 1 and 0 for the control group is same as the counterfactual change in outcomes for the treated group had they not been treated. That is, in absence of the policy, auto sales would have **changed** by the same amount in treatment cities as it did in the control cities. (1 point)*

This assumption is somewhat more plausible than that in a). It assumes that the cities are in the same trend of auto sales. But it is still a strong assumption. If these locations started in period 1 with very different levels of automobile purchases, then there's no reason to assume that the change in purchases over some time interval should counterfactually (absent a policy intervention) be comparable. For example, let's say that consumers in city A bought 1,000 cars in period 1 while consumers in city B bought 100 cars. In period 2, car sales in city A rise to 1,100 and those in city B rise to 110.

(Assume there was no policy intervention in either city in either period). In level terms (# of car sales), sales in city A have risen **by 10 times as much** as in city B. But in proportional terms, they've risen by the same amount (10 percent). Are these two cities on parallel trends? They **are** on parallel trends in **proportional** terms but not in **unit** terms. Which is the “correct” trends comparison—levels or proportions? We really don't know which is correct in advance (perhaps if we had many years of data, we could make an educated guess). This example highlights the fact that difference-in-difference contrasts where the treatment and control groups are not initially comparable must depend upon stronger assumptions about functional form than diff-in-diff comparisons where the treatment and control group are initially comparable. If, for example, cities A and city B had each started with 1,000 car sales in period 1, our conclusion about whether sales rose by more in A versus B in the next period would **not** depend on whether we measured these changes in levels or in proportions. **(1 point)**

- (d) **(2 points)** Write down the estimate that you would use to find the effect of the policy in period 2. Are the identifying assumption different now?

The estimate is simply the same comparison carried out for period 2, comparing changes from period 0. That is:

$$E[Y_{i,t=2} - Y_{i,t=0}|X_i = 1] - E[Y_{i,t=2} - Y_{i,t=0}|X_i = 0]$$

(1 point)

*Note that the identifying assumption is the same: In absence of the policy auto sales would have changed by the same amount between period 0 and period 2 in the treatment cities as it did in the control cities. **(1 point)***

- (e) **(2 points)** If all consumers were like Matt, what would you find in (d)? If you found something different, what would that tell you?

*For Matt, changes in the price of autos in period 1 does not affect his consumption behavior in period 2. Therefore, we would expect a zero effect in part d) if the identifying assumption hold. **(1 point)***

*A non-zero effect here would mean that the identifying assumption in c) do not hold. **(1 point)***

- (f) **(2 points)** If all consumers were like David, would you expect to find no effect of the policy in period 2? Why (not)?

For David, changes in the price of autos in period 1 also affects his consumption of cars in period 2. Therefore, we would expect a non-zero effect in part d) if the identifying assumption hold. As seen from the calculations in part 1, reduction in prices of car in period 1 leads to increased consumption of cars in period 2 as well, for the given preference (Note that whether the effect is positive or negative depends upon the utility functions and are not a general result).

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