14.03/14.003 Problem Set 1 SOLUTIONS
Fall 2014

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Due Monday, 9/15/2012 at 5pm EST

1 Implicit Function Theorem and Envelope Theorem

Consider the function:

$$y = f(x; a) = 1 - ax^2 - a^2 x$$

where a is a parameter.

1. Find the critical point of this function. Denote it with $x^*(a)$.

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ -2ax - a^2 &= 0 \\ x^*(a) &= -\frac{a}{2} \end{aligned}$$

2. Is this point a maximum, a minimum, or neither? (Show why)

   To determine whether this is a maximum or a minimum we check the sign of the second derivative (i.e. Second Order Condition):

$$\frac{d^2y}{dx^2} = -2a$$

   The concavity of this depends on a. If $a < 0$ then $\frac{d^2y}{dx^2} > 0$, implying that we found a minimum. If $a > 0$, then we have a maximum.

3. For the rest of the problem, assume that $a > 0$. Using the critical point of the function $f(x; a)$ found in part 1, find $y^*(a)$, the maximized value of y as a function of a.

$$x^*(a) = -\frac{a}{2}$$
$$y^*(a) = 1 + a^3/4$$

4. Find $\frac{dx^*}{da}$ and $\frac{dy^*}{da}$.

$$\frac{dx^*}{da} = -\frac{1}{2}$$
$$\frac{dy^*}{da} = \frac{3a^2}{4}$$

5. Now use the FOC from (1) and the implicit function theorem to find $\frac{dx^*}{da}$.

$$g(a, x^*(a)) = -2ax^*(a) - a^2 = 0$$
$$\frac{dx^*(a)}{da} = -\frac{g_a(a, x^*(a))}{g_x(a, x^*(a))} = -\frac{-2x^*(a) - 2a}{-2a} = -\frac{a}{-2a} = -\frac{1}{2}$$

6. Use the envelope theorem to find $\frac{dy^*}{da}$. Why does this theorem allow you to simplify your calculations with respect to point 4.?

$$\frac{dy^*}{da} = \frac{\partial y(x^*(a), a)}{\partial a} = -(x^*)^2 - 2ax^* = \frac{3a^2}{4}$$

The envelope theorem allows us to disregard the impact of a change in a on the maximizer x^* since at the optimum $\frac{\partial y}{\partial x^*} = 0$. With respect to part 4, we can avoid substituting $x^*(a)$ to find $y^*(a)$ and its derivative.

2 Constrained Optimization

Consider the problem

$$z(x, y) = x^a y^{(1-a)}$$
$$\max_{x,y} \; z(x, y)$$
$$s.t. \; x + \frac{4}{3}y = 30$$

1. Set up the Lagrangian and find the FOCs.

$$\mathcal{L} = x^a y^{(1-a)} + \lambda \left(30 - x - \frac{4}{3}y\right).$$

The FOCs are

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} a x^{a-1} y^{1-a} - \lambda = 0 \\ (1-a) x^a y^{-a} - \frac{4}{3}\lambda = 0 \\ 30 - x - \frac{4}{3}y = 0 \end{cases}$$

2. Find z^* the maximum, x^* and y^* where the maximum is achieved, and λ the Lagrange multiplier. What is the interpretation of the Lagrange multiplier in this problem?

   Solving the above system, one obtains that $x^* = 30a,\ y^* = 30\frac{3}{4}(1-a),\ \lambda = a(30a)^{1-a}(30\frac{3}{4}(1-a))^a$, *and* $z^* = x^{*a} y^{*(1-a)} = (30a)^a (30\frac{3}{4}(1-a))^{(1-a)}$. *If the constraint is relaxed by one, i.e.* $x + \frac{4}{3}y = 31$, *z can approximately be increased by* λ. Let c be the constraint $x + \frac{4}{3}y = c$. Then we see:

$$\begin{aligned} \frac{dz(x^*, y^*)}{dc} &= \frac{dz(x^*, y^*)}{dx^*}\frac{dx^*}{dc} + \frac{dz(x^*, y^*)}{dy^*}\frac{dy^*}{dc} \\ &= \lambda\frac{dx^*}{dc} + \frac{4\lambda}{3}\frac{dy^*}{dc} \\ &= \lambda(\frac{dx^*}{dc} + \frac{4}{3}\frac{dy^*}{dc}) \\ &= \lambda \end{aligned}$$

   Thus, λ is the implicit shadow price of the constraint $(x + (4/3)\,y = 30)$ expressed in units of z.

3. For the rest of the problem, assume that $a = 0.75$. Set up the dual problem, i.e., minimize the primal constraint function subject to the primal objective function constrained to be equal to the value of z^* obtained in

   For $a = 0.75$, *we get* $x^* = 22.5,\ y^* = 5.625,\ \lambda = 0.5303$ *and* $z^* = 15.9$

$$\begin{aligned} &\min_{x,y} x + \frac{4}{3}y \\ &s.t.\ x^{0.75}y^{0.25} = 15.9 \end{aligned}$$

4. Set up the dual Lagrangian and find the FOCs.

The dual Lagrangian is

$$\mathcal{L}^D = x + \frac{4}{3}y + \lambda^D(15.9 - x^{0.25}y^{0.75}).$$

The FOCs are

$$\begin{cases} \frac{\partial \mathcal{L}^D}{\partial x} = 0 \\ \frac{\partial \mathcal{L}^D}{\partial y} = 0 \\ \frac{\partial \mathcal{L}^D}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} 1 - 0.75\lambda^D x^{-0.25}y^{0.25} = 0 \\ \frac{4}{3} - 0.25\lambda^D x^{0.75}y^{-0.75} = 0 \\ 15.9 - x^{0.75}y^{0.25} = 0 \end{cases}$$

5. Find x^D and y^D where the maximum is achieved and λ^D the Lagrange multiplier in the dual problem. What is the interpretation of the Lagrange multiplier in the dual problem?

   Solving the above system one can get that $x^D = 22.5,\ y^D = 5.625$ *and* $\lambda^D = 1.8856$. *By duality we indeed find* $x^D = x^*,\ y^D = y^*,\ \lambda^D = \frac{1}{\lambda}$. *If the dual constraint is relaxed by one unit, i.e.* $x^{0.75}y^{0.25} = 16.9$, *the dual objective function* $x + \frac{4}{3}y$ *is minimized by approximately an additional* λ^D. *The reasoning for this is similar to part 2. Let* $f = x + \frac{4}{3}y$, $x^{.75}y^{.25} = c$*:*

$$\begin{aligned} \frac{df(x^*, y^*)}{dc} &= \frac{df(x^*, y^*)}{dx^*}\frac{dx^*}{dc} + \frac{df(x^*, y^*)}{dy^*}\frac{dy^*}{dc} \\ &= \lambda^D \frac{dg(x^*, y^*)}{dx^*}\frac{dx^*}{dc} + \lambda^D \frac{dg(x^*, y^*)}{dy^*}\frac{dy^*}{dc} \\ &= \lambda^D(\frac{dg}{dc}) \\ &= \lambda^D \end{aligned}$$

6. What is the relationship between λ and λ^D? Why?

   We found that $\lambda^D = \frac{1}{\lambda}$. *In the primal model, we found that adding an extra unit to* $x + \frac{4}{3}y = 30$ *increased* $x^a y^{(1-a)} = z^*$ *by* λ. *In order to make the connection between the two models, notice that if the dual constraint is tightened by* λ *unit, i.e.* $x^a y^{(1-a)} = z^* + \lambda$, *the dual objective function is increased by* $\lambda \cdot \lambda^D = 1$.

3 Consumer Theory

1. Susan has standard consumer preferences over two goods, Kind Bars (KB) and Red Bull (RB), that satisfy all five axioms of consumer theory. She is indifferent between the

bundles: A=(2KB, 1RB) and B=(1KB, 2RB). She is offered a third bundle C=(1.5KB, 1.5RB). Based on the axioms of consumer theory, what can you infer about Susan's preferences for C relative to A or B. Explain using the axioms.

According to the Diminishing Marginal Rate of Substitution, Susan's preferences will be convex which means that her indifference curves will be convex. Since (1.5KB,1.5RB) lies on or above any convex indifference curves that could connect A and B, we can conclude that Susan either prefers bundle C or is indifferent between all three bundles. Mathematically, we know that Susan's utility function is convex, so $U(1.5, 1.5) \geq 0.5U(2, 1) + 0.5U(1, 2)$.

2. Like Susan, Mahesh has standard consumer preferences over two goods, glasses of orange juice (OJ) and Dunkin bagels (DB). Mahesh's marginal rate of substitution between OJ and DB is 2: that is, at the margin, he would be willing to give up exactly 2 glasses of OJ to get exactly one DB/ Thus, we can say that the total utility of OJ is half the total utility of DB for Mahesh. True, false or indeterminate (and explain)?

   Incorrect. The marginal rate of substitution is well defined, but the cardinal units of utility are not. It would be correct to say that at the margin, the marginal utility of OJ is half the marginal utility of DB. It is incorrect to talk about the total utility of a single good, because Mahesh's total utility is a complicated function of both goods- More OJ could increase his utility for DB, for example. More generally, we care about ordinal utility- the ordering of preferences and the consumer's choice of consumption, not the total utility of a consumer, which is a poorly defined concept. Two utility functions that represent the same preferences but give different utility numbers are functionally identical to us.

3. Consider the preference relation over (x, y) defined by $(x, y)\ ^{p}\ (x', y') \iff x > x'$and $y > y'$. For any pair (x, y) and (x', y'), if it is not the case that $(x, y)\ ^{P}\ (x', y')$ or $(x', y')\ ^{P}\ (x, y)$ then $(x, y)\ ^{I}\ (x', y')$. Does this relation satisfy the first three axioms of consumer preferences?

   This does not satisfy transitivity. $(3, 1)\ ^{I}\ (2, 4)$ *and* $(2, 4)\ ^{I}\ (5, 3)$ *so by transitivity we have* $(3, 1)\ ^{I}\ (5, 3)$ *violates the definition of our preference relation.*

4. Charles has utility function $u = x^{0.5}y^{0.5}$. Does the utility function $v = (xy + 2)^2$ represent his preferences? What about $\mu = \frac{xy+2}{\sqrt{xy}}$? Prove or give counterexamples.

   u and v represent the same preference, because $v = (u^2 + 2)^2$. μ *does not represent the same preferences because they are not related by a monotonic transformation.* $\mu = \frac{u^{2+2}}{u} = u + 2/u$. *Observe that* $\frac{d\mu}{du} = 1 - 2/u$, *and for* $u < 0.5$ *the slope* *is negative. This is not a monotonic transformation and we can conclude that the utility functions have different preferences over some consumption points. For example, utility function u prefers (5,5) over (0.1,0.1), but μprefers (0.1,0.1) over (5,5).* $u(5,5) = 5 < u(0.1, 0.1) = 0.1$*, but* $\mu(0.1, 0.1) = 20.1 > \mu(5,5) = 5.4$*.*

4 Minimum Wages and Employment in the US

Please read the New York Times article (Dube) and the Wall Street Journal article (Neumark). Each writer takes different sides of the minimum wage debate in the US.

1. Which model of the labor market from the ones discussed in class do you think best fit the empirical results on employment effects referred to by both Dube and Neumark? Explain why. [Hint: do the results suggest competitive labor markets for restaurant workers in the US, monopsonistic or something else?] Which assumptions does the model you chose rely on?

   *[**Answer:** mix of competitive and monopsonistic since Dube and Neumark refer to studies with either modest negative effects on employment or no effect. Assumptions for competitive markets are (1) price taking behavior by firms in product and labor markets, so that each firm faces perfectly elastic labor supply; (2) marginal revenue product curve for an individual firm is downward sloping, due to diminishing returns to employment in production function. For monopsonistic labor market: (1) firm has market power so that its demand for labor influences the price at which it can purchase it; (2) all workers receive the same pay; and (3) the marginal revenue product curve for an individual firm is still downward sloping.]*

2. If both Dube and Neumark basically agree on the effect of the minimum wage on employment, what is the basis for their disagreement about the desirability of an increase in the U.S. minimum wage?

   *[**Answer:** The main basis of disagreement is on targeting: Neumark's main argument is that the minimum wage is poorly targeted, for the simple reason that low-wage workers are not necessarily the poorest (e.g. because they are not the primary earner in the household). He argues that the Earned Income Tax Credit would be more effective. On the other hand, Dube argues that the teenage proportion on minimum wages is now low, mitigating some targeting concerns. In addition, Dube argues that there are strong fairness motivations for a minimum wage, citing Camerer and Fehr as an example from the behavioral economics literature that preferences for fairness in transactions matter. In some sense, Dube advances an argument that even if the minimum wage*

is less poorly targeted than EITC (though this is not stated by Dube explicitly), there are deeper, perhaps psychological, reasons why we might still want a higher minimum wage – e.g. because income received as income from a job is treated differently by the recipient from income received as a tax credit.

3. Dube discussed two empirical strategies in particular: one comparing employment in adjacent counties, one using all states in the US. Dube prefers the first – why? Can you give a reason why we might prefer the second approach instead?
   *[**Answer:** Dube is worried about the parallel trends assumption of diff-in-diff – i.e. that employment trajectories of states that raise the minimum wage would be the same as those of states that do not, in the absence of the minimum wage change. Dube thinks this more likely to hold with adjacent counties – e.g. because counties either side of a state border are more similar (than two neighboring states) in terms of local labor market shocks. Alternatively, one might prefer the second approach (comparing across states) if we are worried about spillovers across state borders (e.g., the diff-in-diff comparison could overstate estimates of employment losses if restaurant employers expand just across the border where wages remain low). Note, however, that Dube is arguing that minimum wages do not reduce employment whereas this source of bias would work in the direction of overstating employment losses. Thus, if this bias were present, it would work against Dube's interpretations of the facts (it is therefore a conservative test).]*

4. Now suppose that the US labor market for restaurant workers is perfectly competitive. The labor supply curve is

   $$\ln(w) = {}_s + {}_s \ln(L)$$

   where ${}_s > 0$ (upward-sloping) and the labor demand curve is

   $$\ln(w) = {}_d + {}_d \ln(L)$$

   where ${}_d < 0$ (downward-sloping) and ${}_s < {}_d$ to ensure that an equilibrium exists. What is the equilibrium ln(wage), $\ln(w^*)$? What is equilibrium ln(employment), $\ln(L^*)$? [Hint: you need to set supply of labor equal to demand – solve the 2 equations in 2 unknowns]
   [**Answer:** set supply equal to demand. We get (solving first for ln(wage))

   $$\ln(w^*) = {}_s + \frac{{}_s}{{}_d}\left(\ln(w^*) - {}_d\right)$$

$$\Rightarrow ({}_{d} - {}_{s})\ln(w^*) = {}_{d}\,{}_{s} - {}_{s}\,{}_{d}$$

$$\Rightarrow \ln(w^*) = \frac{{}_{s}\,{}_{d} - {}_{d}\,{}_{s}}{{}_{s} - {}_{d}}$$

Then we can substitute $\ln(w^*)$ into either the supply or demand curve (doesn't matter which since we are in equilibrium in this case). Here we substitute into the supply curve so that: $\ln(L^*) = \frac{\ln(w^*) - {}_{s}}{{}_{s}} = \frac{\frac{{}_{s}\,{}_{d} - {}_{d}\,{}_{s}}{{}_{s} - {}_{d}} - {}_{s}}{{}_{s}} = \frac{{}_{s}\,{}_{d} - {}_{d}\,{}_{s} - {}_{s}({}_{s} - {}_{d})}{{}_{s}({}_{s} - {}_{d})} = \frac{{}_{d} - {}_{s}}{{}_{s} - {}_{d}}$].

5. Now suppose that a minimum wage of $1.5w^*$ is introduced. What condition needs to hold for total earnings ($= wL$) to increase? [Hint: you now need to find the equilibrium wage and employment *after* the minimum wage is introduced, w_2^* and L_2^*, and show what is needed for $w_2^* L_2^* > w^* L^*$ (which is the same as $\ln(w_2^*) + \ln(L_2^*) > \ln(w^*) + \ln(L^*)$)] Give some intuition for why this condition is needed.

   *[**Answer:** total earnings increase if*

$$1.5w^* L_2^* > w^* L^*$$

(where L_2^* is equilibrium employment after the minimum wage is introduced) which is the case if

$$1.5L_2^* > L^*$$

Taking logs (which is a legitimate way of simplifying the expression so long as $L_2^* > 0$), this is the case if

$$\ln(1.5) + \ln(L_2^*) > \frac{{}_{d} - {}_{s}}{{}_{s} - {}_{d}}$$

where $w_2^* = 1.5exp\left(\frac{{}_{s}\,{}_{d} - {}_{d}\,{}_{s}}{{}_{s} - {}_{d}}\right)$ *so that* $\ln(L_2^*) = \frac{\ln(w_2^*) - {}_{d}}{{}_{d}} = \frac{\ln(1.5) + \frac{{}_{s}\,{}_{d} - {}_{d}\,{}_{s}}{{}_{s} - {}_{d}} - {}_{d}}{{}_{d}}$. *So total earnings increase if*

$$\ln(1.5) + \frac{\ln(1.5) + \frac{{}_{s}\,{}_{d} - {}_{d}\,{}_{s}}{{}_{s} - {}_{d}} - {}_{d}}{{}_{d}} > \frac{{}_{d} - {}_{s}}{{}_{s} - {}_{d}}$$

$$\ln(1.5)\,{}_{d}({}_{s} - {}_{d}) + \ln(1.5)({}_{s} - {}_{d}) + {}_{s}\,{}_{d} - {}_{d}\,{}_{s} < {}_{s}\,{}_{d} - {}_{d}\,{}_{s}$$

$$\ln(1.5)(1 + {}_{d})({}_{s} - {}_{d}) < 0$$

$$\Rightarrow {}_{d} < -1$$

since we are given in the question that ${}_{s} > {}_{d}$. Note now that the elasticity of demand *for labor is*

$$\frac{\delta L_d}{\delta w}\frac{w}{L} = \frac{1}{{}_d}$$

It follows then that ${}_d < -1$ *is the same condition that we need the elasticity of demand to be less than one in magnitude – i.e. we need demand to be sufficiently inelastic so that wages rise proportionally more than employment falls. [Note that when demand functions that are log-log, such as* $\ln(w) = {}_d + {}_d \ln(L)$*, the elasticity of demand is constant and equal to* $\frac{1}{{}_d}$*. That is* $\frac{\partial \ln L}{\partial \ln w} = \frac{\partial L}{\partial w} \times \frac{W}{L} = \frac{1}{{}_d}$*] The intuition for the main result is that if labor demand is very elastic, even a small increase in the minimum wage will lead to a large reduction in employment (e.g. because employer can easily substitute to robots), so the total wage bill would fall.]*

5 Causal Inference

An estimated 2 billion people in the world are iron deficient, most of whom are in the developing countries. Iron plays an essential role in energy production in human bodies. Iron deficiency causes anemia (indicated primarily by low levels of hemoglobin) which is associated with greater susceptibility to disease and fatigue. This, arguably, reduces their productivity and may even fetch them lower wages.

The government in your favorite developing country is thinking of introducing a policy of distributing supplemental iron tablets to the iron-deficient population. The government (for some reason) also wants to learn the causal effect of distributing iron on productivity (measured by wages) of the individuals who are iron-deficient.

Denote as $Y_{1i,t}$ as the outcome if the individual i gets the supplement in time t (that is, $X_{i,t} = 1$) and $Y_{0i,t}$ as the outcome if the individual i does not get the supplement in time t (that is, $X_{i,t} = 0$). For this exercise, $t = 0$ represents the time before program implementation and $t = 1$ represents the time after a year of program implementation.

1. Using this notation, and the framework developed in class, write down what the government wants to learn (in terms of the causal effect of interest). Call this effect T^*.

$$T^* = E\left[Y_{1i,t=1} - Y_{0i,t=1} | X_{i,t=1} = 1\right]$$

2. What is the fundamental problem of causal inference in this setting?

   The fundamental problem is that for each individual only one of the potential outcomes $Y_{1i,t=1}$ *or* $Y_{0i,t=1}$ *is observed. The first is observed for those in the treatment group and* *the second is observed for those in the placebo (control) group. That is, for those who are treated, $Y_{0i,t=1}$ is a counterfactual outcome and is never observed. Hence, the T^* cannot be computed without making additional assumptions.*

3. Suppose that the government did a small pilot study among a randomly chosen sample of 2000 adult (aged 15-45) males. The government conducted a hemoglobin screening program (at $t = 0$) and found that 1200 of them were iron-deficient (had hemoglobin levels lower than 12 grams per deciliter (g/dl)). Amongst the iron-deficient population, the government ran a lottery and picked 600 to receive weekly supplemental iron pills. The other 600 received a placebo drug that looked and tasted exactly like the iron pills but did not have any iron supplements. The program continued for a year and at the end of the year, outcomes (hourly wages) were measured for all individuals in the study.

   Suppose the government hired a few consultants to help them analyze the data.

   Denote the post intervention period by $t = 1$. For questions (a) - (c), ignore the 800 that were not-iron deficient.

   (a) The first consultant calculated the following statistic:

   $$E[T_1] = E[(Y_{i,t=1} - Y_{i,t=0})|X_{i,t=1} = 1]$$

   That is, for the individuals treated at $t = 1$, subtract the average outcome at the beginning of the program from the end of the program.

   i. Clearly state (either mathematically, or in very precise words) the assumption(s) under which $E[T_1]$ is an unbiased estimate of T^*.

   $$\begin{aligned} E[T_1] =& E[(Y_{i,t=1} - Y_{i,t=0})|X_{i,t=1} = 1] \\ =& E[(Y_{1i,t=1} - Y_{0i,t=0})|X_{i,t=1} = 1] \\ =& E[(Y_{1i,t=1} - Y_{0i,t=1})|X_{i,t=1} = 1] \\ &+ E[(Y_{0i,t=1} - Y_{0i,t=0})|X_{i,t=1} = 1] \\ =& T^* + E[(Y_{0i,t=1} - Y_{0i,t=0})|X_{i,t=1} = 1] \end{aligned}$$

   (Note that the second line simply substitutes potential outcomes for the realized outcomes. The third step subtracts (from the first term) and adds (to the second term) $E[Y_{0i,t=1}|X_{i,t=1} = 1]$)
   Therefore, for $E[T_1]$ to be an unbiased estimate of T^*, we need that

   $$E[(Y_{0i,t=1} - Y_{0i,t=0})|X_{i,t=1} = 1] = 0$$

That is, if the realized outcome pre-program for those in treatment group is, on average, the same as the potential outcome post-program had they been not treated .

ii. Are these assumptions likely to be satisfied in this context?

This assumption is not likely to be satisfied in this context as wages pre-program is a poor substitute for what the wages would have been absent the program. For instance, wages can be increasing (decreasing) over this period because the economy is doing better (worse).

Another possibility is that workers could be growing more productive over time as they become more experienced.

iii. Give at least one reason this might not be a valid estimate. Explain whether this reason leads to an overestimate ($E[T_1] > T^*$) or an underestimate ($E[T_1] < T^*$)

For both the reasons mentioned above, $Y_{0i,t=1} > Y_{0i,t=0}$ (that is, wages would have increased for the treatment group even if they had not received the treatment), so $E[(Y_{0i,t=1} - Y_{0i,t=0}) | X_{i,t=1} = 1] > 0$. From our calculation in i) it follows that $E[T_1] > T^*$. That is, $E[T_1]$ will be an overestimate. Intuitively, $E[T_1]$ counts increases due to natural productivity increases or national economic trend as the effect of iron-supplements and therefore over-states the effect of the iron supplements.

(b) The second consultant calculated a different statistic:

$$E[T_2] = E[Y_{i,t=1}|X_{i,t=1} = 1] - E[Y_{i,t=1}|X_{i,t=1} = 0]$$

i. Explain verbally what the estimate is computing.

This is simply the difference in outcomes between the treatment ($X_{i,t=1} = 1$) and the placebo (control) ($X_{i,t=1} = 0$) groups measured after the program has completed.

ii. State precisely (ideally, mathematically) the conditions under which $E[T_2]$ is an unbiased estimate of T^*

Doing the similar kind of algebra we did for part a):

$$\begin{aligned}
E[T_2] =& E[Y_{i,t=1}|X_{i,t=1} = 1] - E[Y_{i,t=1}|X_{i,t=1} = 0] \\
=& E[Y_{1i,t=1}|X_{i,t=1} = 1] - E[Y_{0i,t=1}|X_{i,t=1} = 0] \\
=& E[Y_{1i,t=1}|X_{i,t=1} = 1] - E[Y_{0i,t=1}|X_{i,t=1} = 1] \\
& + E[Y_{0i,t=1}|X_{i,t=1} = 1] - E[Y_{0i,t=1}|X_{i,t=1} = 0] \\
=& T^* + E[Y_{0i,t=1}|X_{i,t=1} = 1] - E[Y_{0i,t=1}|X_{i,t=1} = 0]
\end{aligned}$$

Therefore $E[T_2]$ is an unbiased estimate of T^* if

$$E[Y_{0i,t=1}|X_{i,t=1}=1]-E[Y_{0i,t=1}|X_{i,t=1}=0]=0$$

That is, the observed outcome for the control/placebo group at post-program is, on average, the same as the potential outcome for the treatment group at post-program had they been not treated.

iii. Are the conditions satisfied in this context? Explain.
Randomization of individuals into treatment and placebo/control groups insures that these groups were identical (on average) in all observed and unobserved characteristics. Therefore, the control group is a valid counterfactual of what would happen to the treatment group, had they been not treated. Hence, the condition in part ii) is satisfied.

(c) The first consultant wanted to improve on her first estimate. She now computed the following statistic:

$$\begin{aligned} E[T_3] =& E[(Y_{i,t=1}-Y_{i,t=0})\,|X_{i,t=1}=1] \\ & - E[(Y_{i,t=1}-Y_{i,t=0})\,|X_{i,t=1}=0] \end{aligned}$$

i. Explain verbally what the estimate is computing.
She is subtracting the gain in outcomes (wages) between the pre-program and post-program period for the placebo/control group from the gain in outcome between the pre- and post- program for the treatment group.

ii. What is this estimate called?
This estimate is called the difference-in-difference estimate.

iii. State precisely (ideally, mathematically) the conditions under which this estimate will be an unbiased estimate of T^*?

Doing similar algebra as before, we get

$$
\begin{aligned}
E\left[T_3\right] =& E\left[\left(Y_{i,t=1}-Y_{i,t=0}\right)|X_{i,t=1}=1\right]-E\left[\left(Y_{i,t=1}-Y_{i,t=0}\right)|X_{i,t=1}=0\right] \\
=& E\left[\left(Y_{1i,t=1}-Y_{0i,t=0}\right)|X_{i,t=1}=1\right]-E\left[\left(Y_{0i,t=1}-Y_{0i,t=0}\right)|X_{i,t=1}=0\right] \\
& E\left[\left(Y_{1i,t=1}-Y_{0i,t=1}+Y_{0i,t=1}-Y_{0i,t=0}\right)|X_{i,t=1}=1\right] \\
=& -E\left[\left(Y_{0i,t=1}-Y_{0i,t=0}\right)|X_{i,t=1}=0\right] \\
& E\left[\left(Y_{1i,t=1}-Y_{0i,t=1}\right)|X_{i,t=1}=1\right]+E\left[\left(Y_{0i,t=1}-Y_{0i,t=0}\right)|X_{i,t=1}=1\right] \\
& -E\left[\left(Y_{0i,t=1}-Y_{0i,t=0}\right)|X_{i,t=1}=0\right] \\
=& T^* \\
& +E\left[\left(Y_{0i,t=1}-Y_{0i,t=0}\right)|X_{i,t=1}=1\right]-E\left[\left(Y_{0i,t=1}-Y_{0i,t=0}\right)|X_{i,t=1}=0\right]
\end{aligned}
$$

Therefore, $E\left[T_3\right]$ *is an unbiased estimate of* T^* *if*

$$E\left[\left(Y_{0i,t=1}-Y_{0i,t=0}\right)|X_{i,t=1}=1\right]=E\left[\left(Y_{0i,t=1}-Y_{0i,t=0}\right)|X_{i,t=1}=0\right]$$

That is, the observed gain in outcome for the control/placebo group is, on average, the same as the gain in outcomes for the treatment group had they been not treated at all. This is the ***parallel trends*** *assumption: the trends in outcome for the control group has to be a valid counterfactual of the trend in outcome for treatment group.*

iv. Are the conditions satisfied in this context? Explain
Randomization ensures that $E\left[Y_{0i,t=0}|X_{i,t=1}=1\right]=E\left[Y_{0i,t=0}|X_{i,t=1}=0\right]$. *That is, the realized outcome in the pre-program period is the same for both groups. Since, the groups were identical, on average, in every observable and unobservable characteristics, we can expect the control group to be a valid counterfactual of the treatment group post-program as well. This reasoning is same as in part b).*

v. How do you expect this estimate to compare with the one computed in 3b?
As discussed in c.iv) and b.iii), both estimates will be, on average, equal to T^*. *That is, in expectation, they will be the same.*

vi. If you found out that this estimate is slightly different from the one in 3b, what would you infer? Which would you prefer and why?

The estimate is

$$
\begin{aligned}
E\left[T_3\right] =&T^* \\
&+ E\left[\left(Y_{0i,t=1} - Y_{0i,t=0}\right)|X_{i,t=1} = 1\right] - E\left[\left(Y_{0i,t=1} - Y_{0i,t=0}\right)|X_{i,t=1} = 0\right] \\
=&T^* \\
&+ \left(E\left[Y_{0i,t=1}|X_{i,t=1} = 1\right] - E\left[Y_{0i,t=1}|X_{i,t=1} = 0\right]\right) \\
&+ \left(E\left[Y_{0i,t=0}|X_{i,t=1} = 0\right] - E\left[Y_{0i,t=0}|X_{i,t=1} = 1\right]\right) \\
=&E\left[T_2\right] \\
&+ \left(E\left[Y_{0i,t=0}|X_{i,t=1} = 0\right] - E\left[Y_{0i,t=0}|X_{i,t=1} = 1\right]\right)
\end{aligned}
$$

Therefore, any discrepancy in the estimates, arises from the term

$$\left(E\left[Y_{0i,t=0}|X_{i,t=1} = 0\right] - E\left[Y_{0i,t=0}|X_{i,t=1} = 1\right]\right)$$

which will be non-zero if outcomes were not balanced across the treatment and control groups in the pre-treatment period. Therefore, a difference in those two estimates suggests that baseline characteristics were not balanced across the treatment and control group.

*In this scenario, the Diff-in-diff estimate is preferred as it accounts for the discrepancy in outcomes across groups in the pre-program period. The identification assumption of DD estimate - **changes** for both group should be identical in absence, on average, of the program - is more reliable than the assumption for estimate in part b - the **levels** of wages in both groups should be identical, on average, in absence of the program. The latter assumption is more problematic as the outcomes were not balanced across the groups to begin with.*

that control group mimics the counterfactual change for the treatment group