Problem Set 2 Solutions

14.04, Fall 2020

Prof: Robert Townsend

TA: Laura Zhang and Michael Wong[1]

1 Uncertainty and Risk Aversion

A farmer believes that there is a 50-50 chance that the next growing season will be abnormally rainy. His expected utility function has the form

$$U(Y_{NR}, Y_R) = \frac{1}{2}\frac{Y_{NR}^{1-\theta}}{1-\theta} + \frac{1}{2}\frac{Y_R^{1-\theta}}{1-\theta}$$

where Y_{NR} and Y_R represent the farmer's income in the states of "normal rain" and "rainy" respectively.

a) Let $\theta = 1/2$. Suppose the farmer must choose between two crops that promise the following income prospects

Crop	Y_{NR}	Y_R
Wheat	\$28,000	10,000
Corn	19,000	15,000

Which of the crops will he plant?

Solution: If the farmer plants wheat, his expected utility will be

$$28,000^{1/2} + 10,000^{1/2} \approx 267.3$$

Notice that the $\frac{1}{2}$ probabilities cancel with $1 - \theta$.
If the farmer plants corn, his expected utility will be

$$19,000^{1/2} + 15,000^{1/2} \approx 260.3$$

Therefore the farmer chooses to plant wheat because while wheat is more risky, his risk aversion is not that high. He is willing to take on more risk for higher expected utility from a successful wheat crop when it isn't abnormally rain.

b) Suppose the farmer can plant half his field with each crop. This means that in the Y_{NR} state he will earn \$14,000+9,500=23,500 and in the Y_R state he will earn \$5,000+7,500=12,500. Would he choose to do this?

Solution: Expected utility from planting half of his field with each crop would be

$$23,500^{1/2} + 12,500^{1/2} \approx 265.1$$

He would not choose to do this since this is lower expected utility than planting wheat alone. This is because the farmer has low risk aversion, and wheat maximizes expected output $\frac{1}{2}(Y_{NR} + Y_R)$

[1]We greatly appreciate the work put in by Catherine Huang in designing parts of this problem set

c) What mix of wheat and corn would provide maximum expected utility to this farmer?

Solution: The farmer is solving the following optimization problem

$$\max_{x\in[0,1]} \quad (x28,000 + (1-x)19,000)^{1/2} + (x10,000 + (1-x)15,000)^{1/2}$$

The optimum is achieved at $x = 1$ (a corner solution), which we might have guessed at from the earlier answers since the farmer has low risk aversion and therefore would prefer to plant only wheat.

d) Let $\theta = 3/4$. How would your answer to (c) change in this case? Give an intuitive reason why a higher risk aversion parameter changes his decisions.

Solution: Now the farmer has higher risk aversion, which would suggest that he would want to plant less wheat, which is a very risky crop. The optimization problem is now

$$\max_{x\in[0,1]} \quad (x28,000 + (1-x)19,000)^{1/4} + (x10,000 + (1-x)15,000)^{1/4}$$

Note the scalar multiple does not affect the optimal solution of x. The optimal x is $\approx .694$

2 Pareto Optimality

Suppose individuals denoted by i have happiness functions that only depend on one's own consumption, called $u_i(x_i)$ where x_i is a vector of one's own consumption. However, individuals are altruists in that they care about other people's happiness also. Let I be the total number of people. Each person's overall utility function is $U_i(x_1, \ldots, x_I)$ where U_i has the form

$$U_i(x_1, \ldots, x_I) = U_i(u_1(x_1), u_2(x_2), \ldots, u_I(x_I))$$

Assume that U_i is strictly increasing in all the arguments. Here, individual i's utility depends on all the other people's happiness functions $u_1(x_1), u_2(x_2), \ldots, u_I(x_I)$.

a) Show that if consumption allocation $x = (x_1, \ldots, x_I)$ is Pareto optimal under altruistic utility function $U_i(\cdot)$, then allocation x is also Pareto optimal under the individualistic utility function $u_i(\cdot)$ (where individuals only care about their own happiness).

Solution: We can prove the contrapositive that if x is not Pareto optimal under $u_i(\cdot)$, then x is not Pareto optimal under $U_i(\cdot)$. If x is not Pareto optimal under $u_i(\cdot)$, then there exists y that Pareto dominates x under u_i. Then $u_i(y_i) \geq u_i(x_i)$ for all i and $u_i(y_i) > u_i(x_i)$ for some i. Thus $U_i(y) > U_i(x)$ for every i since U_i is strictly increasing in the $u_1, \ldots, u_I$ values. This proves that y Pareto dominates x under U_i and therefore x is not Pareto optimal under U_i.

b) Does this mean that a community of altruists can use standard competitive markets to attain Pareto optimality? (Optional)

3 Dynamic Programming

Consider a farmer who lives for T periods. In each period t, the farmer chooses how much to consume (c_t) and how many seeds to plant to be available in the next period (K_{t+1}). In particular, consumption at date t is given by

$$c_t = \alpha K_t - K_{t+1}$$

where $\alpha > 0$ is the yield to seed ratio. The farmer then solves the problem

$$\max_{\{K_{t+1}, c_t\}_{t=1}^{T}} \quad \sum_{t=1}^{T} \beta^{t-1} u(c_t) \tag{1}$$

$$s.t.$$

$$c_t = \alpha K_t - K_{t+1}$$

$$c_t \geq 0 \tag{2}$$

where $u(c_t)$ is the utility the farmer gets from consumption in period t, $\beta \in (0, 1)$ and $K_1 > 0$ is given exogenously (e.g. seeds the farmer inherited when he was born). Let $\{K^*_{t+1}, c^*_t\}_{t=1}^{T}$ solve the above problem. That is, the farmer picks a sequence of consumption and seed-planting decisions in each period to solve the above problem.

a) Argue that, regardless of the value of T, we will always have that $K^*_{T+1} = 0$.

Solution: If the farmer's solution involved $K^*_{T+1} > 0$, then by decreasing K_{T+1} they could increase c_T (without affecting consumption in any previous period). So it cannot be an optimum. Intuitively, planting is only useful to the farmer because it enables them to consume more next period, and this benefit is irrelevant in the final period.

b) Now let's look at the case where $T = 2$. Use the fact that $K^*_3 = 0$ and the constraints to eliminate c_t and write the farmer's problem as an optimization problem in one variable only, K_2. What constraint must K_2 satisfy?

Solution: Because $K^*_3 = 0$ we have immediately $c_2 = \alpha K_2$. Furthermore $c_1 = \alpha K_1 - K_2$. So we can substitute these into the utility function to get that the farmer maximizes

$$u(\alpha K_1 - K_2) + \beta u(\alpha K_2)$$

Because consumption must be non-negative in each period, K_2 must satisfy $0 \leq K_2 \leq \alpha K_1$.

c) Suppose the farmer's per-period utility takes the form $u(c_t) = c_t$. Solve for the farmer's optimal planting decision, K^*_2. (Hint: there will be two cases, depending on the values of α and β). What is consumption in each period?

Solution: Now the farmer's problem is

$$\max_{K_2 : 0 \leq K_2 \leq \alpha K_1} \alpha K_1 - K_2 + \alpha\beta K_2$$

If $\alpha\beta > 1$, they maximize this by setting $K_2 = \alpha K_1$, i.e. planting everything in the first period. In this case consumption in each period is $c_1 = 0$, $c_2 = \alpha^2 K_1$. If $\alpha\beta < 1$, they maximize this by setting $K_2 = 0$, i.e. by eating everything in the first period. Consumption is $c_1 = \alpha K_1$ and $c_2 = 0$.