Problem Set 3 Solutions

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1 Risk Sharing

Ridhi and Neha are neighbors who farm wheat for a living. In each of 30 periods, they grow, harvest, and eat wheat. They are permitted to gift wheat to their neighbor, but they may not save wheat between periods or throw wheat away. The two farmers have CRRA utility functions, such that if one of them has risk aversion parameter θ and consumes c bushels of wheat, that farmer gets

$$U(c) = \frac{c^{1-\theta} - 1}{1 - \theta} \tag{1}$$

units of utility in the given period. Ridhi is risk adverse, with $\theta_{Ridhi} = 0.999$ and Neha is nearly risk neutral, with $\theta_{Neha} = 0.001$.[2] In the "Risk Sharing Simulation.xlsx" model, they each grow on average 28 bushels of wheat when output is High. When output is Low, they each grow on average 10 bushels of wheat.

Notes on Excel program: You need to enable macros when you open the Excel document to use the formulas. If you get a compile error, you may be using an earlier version of Excel that does not have the functions needed to run the program. Go to MIT IS&T to download the 2019 version of Excel.

1.1 Shared Shock, Different Risk Aversion

Open "Risk Sharing Simulation.xlsx" and enable macros. In this problem set you will analyze contracts in two ways: (1) by using the simulation as a rough approximation of risk sharing results, and (2) by calculating the expected utilities exactly.

Suppose the weather determines whether harvests are High or Low, meaning Ridhi and Neha experience the same shocks to their output. If the weather is good, both enjoy High output. If the weather is bad, both suffer Low output. The weather is good 1/2 the time and bad 1/2 the time.

a) Ridhi and Neha's friend Rob proposes a contract in which Ridhi gives Neha 9 bushels of wheat when the weather is good, and Neha gives Ridhi 9 bushels of wheat when the weather is bad. Use the "Randomize State and Yield" button to repeatedly simulate 30 periods of consumption and utilities in

[1]We greatly appreciate the work put in by Catherine Huang in designing parts of this problem set.

[2]As θ_{Neha} approaches 0, $U(c) \approx c - 1$. As θ_{Ridhi} approaches 1, $U(c) \approx \ln(c)$.

the Excel spreadsheet. Is Ridhi's simulated mean utility typically higher under autarky or under Rob's risk-sharing contract? Answer the same for Neha. (A few sentences will suffice.)

Solution: Under almost all draws, Ridhi has higher average utility under risk sharing than in autarky. Neha's utility fluctuates above and below her autarky utility, and is on average about the same as in autarky.

b) Approximate Ridhi and Neha's utilities by letting $U_{Ridhi}(c) = \ln(c)$ and $U_{Neha}(c) = c - 1$. Calculate each farmer's expected utility first in autarky, then under the contract. Will Ridhi will accept Rob's risk-sharing contract? Will Neha? Provide intuition for why in each case.

Solution: Expected utilities follow

	Autarky	Risk Sharing
Ridhi	2.82	2.94
Neha	18	18

Ridhi has higher expected utility under risk sharing, since expected consumption is unchanged but its variance is reduced. Neha has the same expected utility under risk sharing, since Neha is risk-neutral and expected consumption is the same under both arrangements. They will both accept the contract.

1.2 Shared Shock, Same Risk Aversion

a) Now suppose output level still depends on the weather, but Neha's risk aversion increases such that $\theta_{Neha} = 0.999$ also. Using simulations from the Excel spreadsheet, describe whether Neha's simulated mean utility is typically higher under autarky or under Rob's risk-sharing contract. Is this different from your answer that in 1.1(a)? Why?

Solution: Ridhi is better off under the contract as before, but Neha is now worse off. Intuitively, the contract negatively impacts Neha, since she is risk averse and it increases the variance of her consumption without changing her average consumption.

b) Approximate Ridhi and Neha's utilities by letting $U_{Ridhi}(c) = \ln(c)$ and $U_{Neha}(c) = \ln(c)$. Use expected utility theory to explain whether Ridhi and Neha will both accept the contract. Provide intuition.

Solution: Expected utilities follow

	Autarky	Risk Sharing
Ridhi	2.82	2.94
Neha	2.82	1.81

The contract increases Ridhi's expected utility and decreases Neha's expected utility. Neha will not sign the contract. Previously, Neha was willing to take on Ridhi's risk because Neha was risk neutral. Now Neha is just as risk averse as Ridhi. Neha no longer finds it worth taking on Ridhi's risk.

c) Is there any contract that Ridhi and Neha would both accept? Why?

Solution: No there is not. Suppose a contract exists such that both Ridhi and Neha are willing to sign. This contract must involve a positive transfer from Ridhi to Neha in one weather state, and a positive transfer from Neha to Ridhi in the other state. Otherwise, the sole giver would not accept the contract. We solve the following system of inequalities:

$$\ln(28-b)+\ln(10+a) \geq \ln(28)+\ln(10)$$
$$\ln(28+b)+\ln(10-a) \geq \ln(28)+\ln(10)$$

Equivalently,

$$(28-b)(10+a) \geq 280 \Rightarrow 28a-10b-ab \geq 0$$
$$(28+b)(10-a) \geq 280 \Rightarrow -28a+10b-ab \geq 0$$

a and b must be such that $-2ab \geq 0$. This would require for exactly one of a, b to be negative, or for at least one of a, $b = 0$. Neither option constitutes a viable contract. Thus, there is no contract that both Ridhi and Neha will sign. *Based on the math above, we can take this conclusion a step further: there is no contract that both Ridhi and Neha will sign for any endowments in the Low and High output states, as long as Ridhi and Neha get the same output as each other in each state.

1.3 Idiosyncratic Shock, Same Risk Aversion

Suppose $\theta_{Ridhi} = \theta_{Neha} = 0.999$ still, but now shocks are idiosyncratic: the weather doesn't matter, but each year, a swarm of Hessian flies attacks Ridhi's farm with probability 1/2. The same swarm may attack Neha's farm with independent probability 1/2. If a farm is attacked, that farmer's output is Low. If a farm is not attacked, that farmer's output is High. Rob also changes the contract. If Ridhi has High output, she gives Neha 6 bushels regardless of Neha's output in the same period. If Neha has High output, she gives Ridhi 7 bushels regardless of Ridhi's output in the same period. Farmers do not give anything to their neighbor if they experience Low output.

a) Using simulations from the Excel spreadsheet, explain whether Neha's simulated mean utility is typically higher under autarky or under Rob's risk-sharing contract and why. Do the same for Ridhi.

Solution: Answer: In most draws, Ridhi and Neha are both better off under the contract.

b) Approximate Ridhi and Neha's utilities by letting $U_{Ridhi}(c) = \ln(c)$ and $U_{Neha}(c) = \ln(c)$. Use expected utility theory to explain whether Ridhi and Neha will both accept the contract.

Expected utilities follow

	Autarky	Risk Sharing
Ridhi	2.82	2.90
Neha	2.82	2.85

Both Ridhi and Neha have higher expected utility under risk sharing. They will accept the contract.

1.4 Idiosyncratic Shock, Different Risk Aversion

Now, let $\theta_{Ridhi} = 0.999$ and $\theta_{Neha} = 0.001$. Consider a contract that takes the same form as the one in Question 3: if Ridhi has High output, she gives Neha r bushels regardless of Neha's output in the same period. If Neha has High output, she gives Ridhi n bushels regardless of Ridhi's output in the same period.

With risk neutral utility, Neha will reject the contract in Question 3. Given $\theta_{Ridhi} = 0.999$ and $\theta_{Neha} = 0.001$, is there a contract both farmers would accept? Explain.

Solution: Neha is willing to accept any contract in which Ridhi pays Neha more than Neha pays Ridhi. Take a contract in which Neha only pays Ridhi 7 bushels when Neha has High output, while Ridhi pays Neha 8 bushels when Ridhi has High output. Intuitively, under this contract Ridhi pays Neha for Neha to take on Ridhi's risk. Such a contract makes much more sense. Indeed, the table of expected utilities shows that both farmers are better off under this contract:

	Autarky	Risk Sharing
Ridhi	2.82	2.85
Neha	18	18.5

2 Contracting with Asymmetric Information

A monastery (M) is endowed with W units of wealth and a villa V is endowed with θ units of wealth, where θ is a random variable taking on the following values:

$$\theta = \begin{cases} H & \text{w. prob. } 0.5 \\ L & \text{w. prob. } 0.5 \end{cases}$$

Assume $W > H > L$. The monastery can send a transfer τ to the villa. (We allow the possibility of $\tau < 0$, i.e. the villa pays the monastery.) The monastery's utility is $W - \tau$ and the villa's utility is $u(\theta + \tau)$ where $u(\cdot)$ is strictly increasing and strictly concave.

There is a social planner, who chooses the transfer to maximize a λ-weighted sum of the monastery's and villa's *expected* utilities.

a) Suppose for now that the social planner can observe the realization of θ and can decide on a transfer $\tau(\theta)$ after they have done so. Set up the social planner's optimization problem.

   Solution: The social planner solves

$$\max_{\tau(H),\tau(L)} \lambda(0.5(W-\tau(H))+0.5(W-\tau(L)))+(1-\lambda)(0.5u(H+\tau(H))+0.5u(L+\tau(L)))$$

   which simplifies to

$$\lambda(W-\frac{\tau(H)+\tau(L)}{2})+\frac{1-\lambda}{2}(u(H+\tau(H))+u(L+\tau(L)))$$

b) Derive the first-order conditions and show that the social planner optimally chooses the transfer so that the villa's consumption $(\theta+\tau(\theta))$ does not depend on θ.

   Solution: The first-order conditions give

$$u'(H+\tau(H)) = \frac{\lambda}{1-\lambda}$$
$$u'(L+\tau(L)) = \frac{\lambda}{1-\lambda}$$

   u is strictly concave, so this implies $H+\tau(H) = L+\tau(L)$.

Now suppose the social planner cannot observe θ. Instead, only the villa sees the value of θ. The social planner relies on the villa to make a report of θ, $\tilde{\theta}$, to her, and then chooses a (non-random) transfer $\tau(\tilde{\theta})$.

c) Suppose the social planner simply assumes the villa's report is always truthful, so that she selects the $\tau(H)$ from part b) when the villa reports $\tilde{\theta} = H$ and the $\tau(L)$ from part b) when the villa reports $\tilde{\theta} = L$. What will the villa optimally report:

   i. When $\theta = H$?
   ii. When $\theta = L$?

   Solution: Because $H > L$ and $H+\tau(H) = L+\tau(L)$, we have $\tau(L) > \tau(H)$. Hence the villa will always report that $\theta = L$, whether or not it actually is.

d) Write down the incentive compatibility constraints that the social planner's transfer scheme $\tau(\tilde{\theta})$ must satisfy if the villa is to tell the truth. What do these constraints imply about $\tau(H)$ and $\tau(L)$?

   Solution: The constraints are

$$u(H+\tau(H)) \geq u(H+\tau(L))$$

$$u(L + \tau(L)) \geq u(L + \tau(H))$$

The first implies $\tau(H) \geq \tau(L)$ and the second implies $\tau(L) \geq \tau(H)$. So we must have $\tau(H) = \tau(L)$. The social planner can achieve no risk-sharing with a deterministic transfer.

Now, suppose the social planner can offer a lottery as a transfer. That is, the transfer can take on K values $\{\tau_1, \ldots, \tau_K\}$ and the social planner can choose the probability of each value conditional on the report, $\{\pi(\tau_k|\tilde{\theta})\}_{k=1}^K$.

e) Write down the social planner's optimization problem now, including the new incentive compatibility constraints.

Solution: The social planner solves

$$\max_{\tau(H),\tau(L)} \lambda \left(0.5 \sum_{k=1}^{K} \pi(\tau_k|H)(W - \tau_k) + 0.5 \sum_{k=1}^{K} \pi(\tau_k|L)(W - \tau_k) \right)$$

$$+(1-\lambda) \left(0.5 \sum_{k=1}^{K} \pi(\tau_k|H) u(H + \tau_k) + 0.5 \sum_{k=1}^{K} \pi(\tau_k|L) u(L + \tau_k) \right)$$

subject to

$$\sum_{k=1}^{K} \pi(\tau_k|H) u(H + \tau_k) \geq \sum_{k=1}^{K} \pi(\tau_k|L) u(H + \tau_k)$$

$$\sum_{k=1}^{K} \pi(\tau_k|L) u(L + \tau_k) \geq \sum_{k=1}^{K} \pi(\tau_k|H) u(L + \tau_k)$$

f) Discuss briefly how one could, in practice, solve this problem given a functional form for $u(\cdot)$ (but there is no need to actually solve it).

Solution: This is a linear programming problem and we can solve it using linear optimization solvers in, for instance, Matlab.

3 Contracting with Moral Hazard

Consider a principal (P) and an agent (A). A can exert an effort level $e \in \{0, 1\}$, which yields some profit πe to the principal, where $\pi \in \mathbb{R}$. P can make a transfer to A denoted by $t \in \mathbb{R}$. The player's payoffs are given by

$$\begin{aligned} u_A(e, t) &= t - e \\ u_P(e, t) &= \pi e - t \end{aligned}$$

A social planner can choose a **deterministic** allocation rule (e, t) to implement. In this question, we will focus on the case where the social planner always wants to maximize the unweighted sum of the two player's payoffs.

a) Suppose that the social planner can directly choose A's level of effort e. Write down the planner's problem to find e^{FB} and t^{FB} that maximizes the unweighted sum of the two player's utilities. Under what condition is $e^{FB} = 1$?

Solution: The planner solves

$$\max_{e,t}(t - e) + (\pi e - t)$$

Note that this reduces to

$$\max_{e,t}(\pi - 1)e$$

Therefore, we have that $e^{FB} = 1$ if $\pi > 1$. The agent exerts effort if the gain from effort is larger than the cost of effort.

Since t does not enter the objective function above, any t would solve the planner's problem.

b) Now suppose that the social planner must choose an effort level e that A is willing to perform. Write down the planner's problem to find e^{MH} and t^{MH} that maximize the sum of the two player's utilities, subject to A's incentive compatibility constraints, which requires that the agent is willing to exert the chosen level of effort.

Solution: The planner will solve

$$\max_{e,t}(\pi - 1)e$$

subject to

$$t - e \geq t - \tilde{e} \text{ for } \tilde{e} \in \{0, 1\} \tag{2}$$

The constraint says that the agent's utility from the chosen level of effort must be weakly greater than the agent's utility from any other level of effort.

c) Is it ever possible to have $e^{MH} = 1$? What if we allow for lotteries over transfers?

Solution: No it is not possible. Rearranging (2), we have that $e^{MH} \leq 0$.

Allowing for lotteries does not change this incentive compatibility constraint.