Problem Set 5 Solutions

14.04, Fall 2020
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1 Walrasian Equilibrium and Welfare Theorems

There are two consumers, both with Leontief preferences for two goods x, y in an exchange economy. The consumers' utility functions are

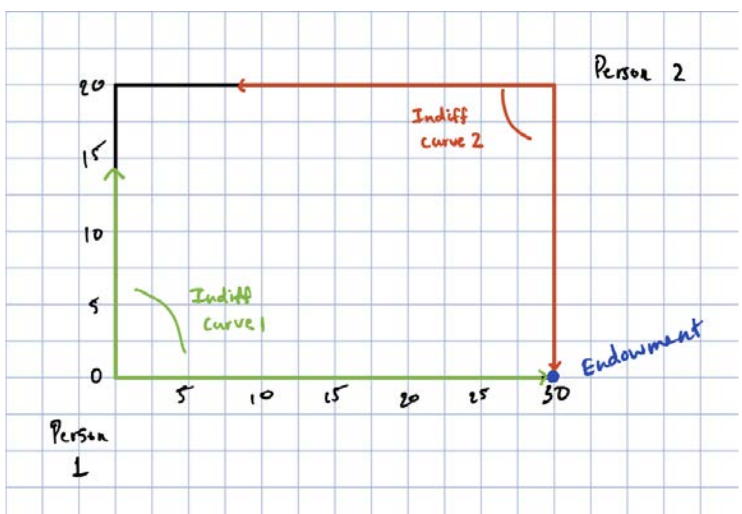
$$U_1(x_1, y_1) = \min\{x_1, y_1\}$$

$$U_2(x_2, y_2) = \min\{4x_2, y_2\}$$

and the endowments are such that consumer 1 has only 30 units of good x, and consumer 2 has only 20 units of good y. Assume that prices $p_x, p_y \in [0, \infty)$ and that agents can only have non-negative consumption.

a) Draw an Edgeworth box with the endowment point and the two agents' indifference curves going through the endowment point.

Solution:



b) What are the competitive equilibria in this economy? *Hint:* prices can be 0. Redraw these competitive equilibria on your Edgeworth box. Are the equilibria also Pareto optimal allocations?

Solution: The indifference curves are right angles with vertices at $y_1 = x_1$ and $y_2 = 4x_2$, and the consumers can maximize utility by consuming at the vertices for any budget line with positive prices for both goods. We can draw the indifference curves of each consumer in an Edgeworth box, and it is clear there that the lines representing the vertices never intersect. There cannot be an equilibrium with both consumers consuming positive amounts of both goods.

Suppose that prices are $p_x = 0, p_y > 0$, so movements on the budget line along the x-axis do not change the allocation along the y-axis. Then starting from the endowment, it is possible to increase and maximize 2's utility without changing consumer 1's utility by consuming at the allocation $(x_1, y_1) = (25, 0), (x_2, y_2) = (5, 20)$. This allocation is equivalent utility-wise to any allocation in the set $(x_1, y_1) = (30 - s, 0), (x_2, y_2) = (s, 20)$ for $s \in [5, 30]$. These are competitive equilibria.

Suppose that prices are $p_x > 0, p_y = 0$ so movements on the budget line along the y-axis do not change the allocation along the x-axis. Starting from the endowment with movements along the y-axis, it is not possible to achieve an allocation where consumer 1 is maximizing utility.

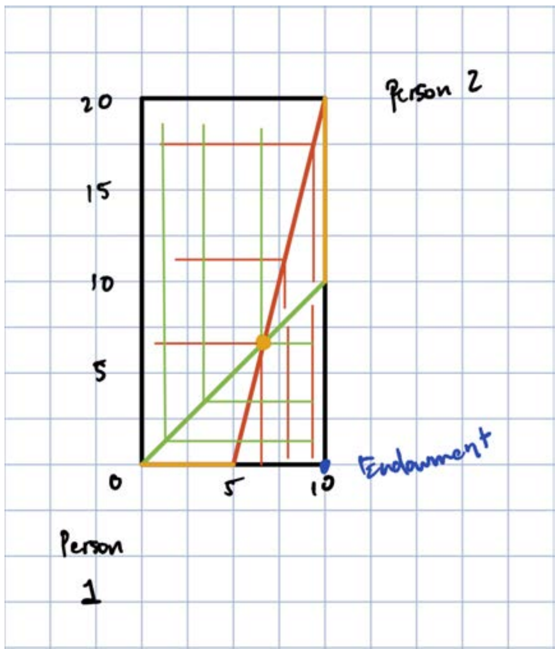
All competitive equilibria are Pareto optimal from the First Welfare theorem. We can see this also in the Edgeworth box since the two indifference curves are tangent to each other at these allocations, so neither agent can be made better off.

c) Suppose that consumer 1 instead has an endowment of 10 units of good x and again no units of good y. What are the competitive equilibria in this case? Draw these competitive equilibria on a new Edgeworth box. Be sure to find all the equilibrium allocations.

Solution: With these endowments, the vertices of the right-angle indifference curves for the two consumers do intersect. Therefore, there is an allocation with positive prices for both goods where both consumers are maximizing utility. This allocation occurs at $(x_1, y_1) = (20/3, 20/3), (x_2, y_2) = (10/3, 40/3)$ with the prices p_x, p_y following the ratio $p_x/p_y = 2$

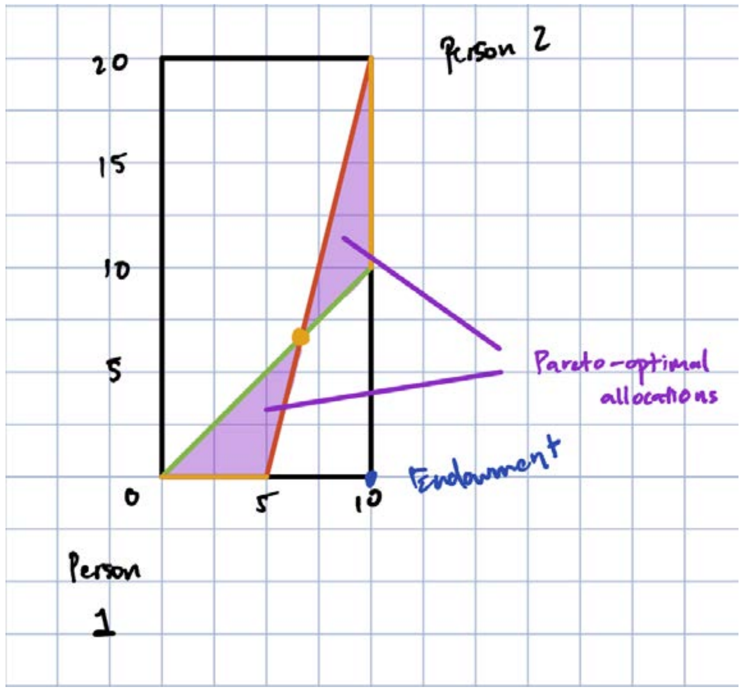
As in the above question, when prices are zero, there are also equilibria. With $p_x = 0, p_y > 0$, the set of equilibria follows $(x_1, y_1) = (10 - s, 0), (x_2, y_2) = (s, 20)$ for $s \in [5, 10]$.

With $p_x > 0, p_y = 0$, the set of equilibria follows $(x_1, y_1) = (10, s), (x_2, y_2) = (0, 20 - s)$ for $s \in [10, 20]$.



d) Now suppose that there is a social planner who can pick an endowment allocation vector $\omega_i = (\omega_{xi}, \omega_{yi}) \geq 0$ for agents $i \in \{1, 2\}$. The social planner is restricted by the economy's resource constraints. In particular, suppose we have the total resources as in (c) so that $\omega_{x1} + \omega_{x2} = 10, \omega_{y1} + \omega_{y2} = 20$. On your Edgeworth box from part c) show the set of all allocations that can be supported as a competitive equilibrium for some endowment allocation.

Solution: Given that preferences are locally non-satiated, we can apply FWT and conclude that any equilibria must be Pareto optimal. Using the fact that preferences are (weakly) convex, we can apply SWT and conclude that there is some endowment allocation for every Pareto optimal allocation, that makes that allocation a competitive equilibrium. Thus, the set we are looking for is simply the set of all Pareto optimal allocations. From the Edgeworth box we can see that this consists of all allocations between the lines representing the vertices of the two agents' indifference curves.



2 Existence

Consider an exchange economy with three individuals and two goods. Each individual has the utility function

$$U^i(x, y) = x^2 + y^2$$

and has an endowment $(1, 1)$.

a) Are these preferences convex?

   Solution: No.

b) Calculate the demands of each individual when they face prices $(1, p)$.

   Solution: With strictly convex utility the individual puts all their money into whichever good is cheaper. So they demand $(1 + p, 0)$ when $p > 1$, and $(0, (1 + p)/p)$ when $p < 1$. When $p = 1$ they demand either $(1 + p, 0)$ or $(0, (1 + p)/p)$.

c) Show that there cannot be a Walrasian equilibrium with $p \neq 1$.

   Solution: If $p < 1$ then everyone demands only x, and if $p > 1$ then everyone demands only y. In both cases markets cannot clear.

d) Show that in fact, there does not exist any Walrasian equilibrium for this economy[1]. Where does Negishi's proof break down?

[1] Assume lotteries are not possible - each consumer consumes a non-random amount.

Solution: The only remaining possibility is an equilibrium with $p = 1$. When $p = 1$, each person demands either $(0, 2)$ or $(2, 0)$. But there is no way to make these demands add up to the total endowment of $(3, 3)$. Negishi's proof breaks down because preferences are non-convex so we cannot use the second welfare theorem to implement any Pareto-optimal point as an equilibrium (note a Pareto-optimal allocation will still exist here).

3 Quasi-Linear Utility and Gorman Form

Consider a consumer with quasi-linear preferences over two consumption goods X and Y given by $U(X, Y) = X + \ln(Y)$. Assume that the consumer has wealth w and that consumption of both goods must be weakly positive, i.e. $(X, Y) \in \mathbb{R}^2_+$.

a) Write down the utility maximization problem for the consumer and the first-order conditions for the optimal consumption bundle (X, Y).

Solution: We have that the consumer solves

$$\begin{aligned}
&\max_{X,Y} X + \ln(Y) \\
&\text{s.t.} \\
&p_X X + p_Y Y \leq w \\
&X \geq 0 \\
&Y \geq 0
\end{aligned}$$

The first-order conditions give us that

$$\begin{aligned}
1 &= \lambda p_X - \mu_X \\
\frac{1}{Y} &= \lambda p_Y - \mu_Y
\end{aligned}$$

where λ is the lagrange multiplier on the budget constraint and μ_X, μ_Y are the non-negativity constraints on X and Y respectively.

b) Show that the consumer's demand functions are given by

$$\begin{aligned}
X(p_X, p_Y, w) &= \frac{w}{p_X} - 1 \\
Y(p_X, p_Y, w) &= \frac{p_X}{p_Y}
\end{aligned}$$

You can assume an interior solution.

Solution: Let (X^*, Y^*) solve the problem without the non-negativity constraints. The FOCs from part a) implies that $Y^* = \frac{p_X}{p_Y}$. Given this, we must have that

$$\begin{aligned}
X^* &= \frac{w - p_Y \frac{p_X}{p_Y}}{p_X} \\
&= \frac{w}{p_X} - 1
\end{aligned}$$

c) Compute indirect utility $V(p_X, p_Y, w)$.

Solution: We have that

$$V(p_X, p_Y, w) = \frac{w}{p_X} - 1 + \ln\left(\frac{p_X}{p_Y}\right)$$

d) What does it mean for an individual's preferences to satisfy Gorman form? Does the consumer's utility satisfy Gorman form?

Solution: A consumer's preferences satisfy Gorman form if indirect utility takes the form

$$V(p, w) = a(p) + b(p)w$$

The consumer's utility does satisfy Gorman form with $a(p) = \ln\left(\frac{p_X}{p_Y}\right) - 1$ and $b(p) = \frac{1}{p_X}$.

14.04: Intermediate Microeconomic Theory
Fall 2020