

# Lecture 4: The Solow Model

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# This lecture: the determinants of capital

- Production model:  $k$  and  $A$  matter for income differences.
- Next several lectures: endogenizing  $k$

Last time: role of  $k$  in an “accounting” sense. We will go deeper now.

- What determines a country's capital stock? How does it evolve over time? How can we increase the capital stock?
- How does endogenizing  $k$  change our understanding of the sources of income differences across countries?
- Our starting point is the Solow growth model ...

# Outline

## 1. Basics of the Solow growth model

## 2. Characterization of equilibrium

Steady state

Transitional dynamics

Summary

## 3. Testable implications

Capital accumulation & growth

Conditional convergence

Divergence

# Introduction to the Solow model

- Developed by Robert (Bob) Solow from MIT and Trevor Swan. Many implications for growth. Solow won the Nobel prize.
- Solow's key insight was to utilize the neoclassical production function (that we saw the last time) in a growth context.
- In his work he combined this with (fixed, not necessarily optimized) saving behavior to obtain a theory of how  $k$  evolves over time.
  - Question: what are the effects of savings and capital accumulation on long-run growth?
  - The Ramsey growth model—which we will see later—thinks about *optimal* savings.

# Production

- Time is discrete,  $t \in \{0, 1, 2, \dots\}$ . Could be a year, a quarter, etc.  
Our models so far (RC and production model) have been static. This is the first dynamic model.
- At each period, the economy has total capital  $K_t$  and labor  $L_t$ .
- Given  $K_t$  and  $L_t$ , output is produced according to the technology

$$Y_t = F(K_t, L_t)$$

- We will assume that the production function is neoclassical, exactly as last time.
  - Output can be used for two purposes: consumption and investment. This will be the key margin driving everything in the model.
- Key difference from last lecture: now we don't just take a snapshot at a single  $t$ , but we ask how the economy—i.e.,  $Y_t$ ,  $K_t$ ,  $L_t$ —evolves over time

# Production

- There are firms that hire capital and labor, and households that own and supply these factors, and they meet in competitive markets, like last time
- But today: assume a “central planner,” or a “big family” representing the entire economy, that directly runs the firms and produces  $F(K_t, L_t)$ 
  - Why? 1st welfare th'm—competitive eq'm allocation = planner outcome, so we are confident a competitive market would generate this allocation.
  - We will formally revisit the equilibrium vs. planner comparison when we study the optimal consumption-savings choice (= Ramsey growth model).
  - In addition to organizing production to deliver output  $Y_t = F(K_t, L_t)$ , the planner also decides how much to consume and how much to save.
- Next few slides: evolution of  $L_t$  and  $K_t$  (and thus  $Y_t$  as  $Y_t = F(K_t, L_t)$ )

# Labor grows at a constant rate

- We take  $L_t$  as (largely) exogenous to the model.
- In particular, suppose  $L_t$  evolves according to:

$$L_t = (1 + n)L_{t-1} = (1 + n)^t L_0.$$

Here,  $n$  captures population growth.

- In contrast,  $K_t$  is determined endogenously in the model.

# Capital changes due to investment

- In particular, output is split between consumption and investment:

$$Y_t = C_t + I_t.$$

This is the resource constraint of the economy.

Compared to our earlier models, we now have two uses for output: consumption or investment.

- A constant fraction of capital,  $\delta$ , depreciates after production. Thus, the law of motion of capital is given by:

$$K_{t+1} = (1 - \delta)K_t + I_t.$$

- Everything so far was the exogenous part of the model: population growth, resource constraints, technology = production function.
- Missing part: what determines the split between consumption and investment?

# Key assumption: constant savings rate

- We will for now assume that this decision is really simple—the economy (= the planner) saves a fixed fraction of output:

$$I_t = sY_t = sF(K_t, L_t)$$

- Here  $s \in (0, 1)$  is the constant savings rate. Rest is consumed, so  $C_t = (1 - s)Y_t$ .
  - Again, this could be generated via households and firms interacting. But for now assume the planner just dictates everything. Will revisit.
- Law of motion of capital can now be written as:

$$K_{t+1} = (1 - \delta)K_t + sF(K_t, L_t)$$

- Solow model [today] vs. Ramsey model [soon]
  - Only difference: in Ramsey, the savings decision will be endogenized—it will follow from optimizing behavior.
  - Many of the conclusions from the Solow model will turn out to be robust. Key value-added of the Ramsey growth model: understand how incentives and policies matter for savings.

## Law of motion in intensive form

- Let's work a bit more on the model equations. Easier to do so in intensive (per labor) form. Dividing each side of the law of motion with  $L_t$ , we have:

$$\frac{K_{t+1}}{L_t} = (1 - \delta)k_t + sf(k_t).$$

- Let us multiply both parts with  $\frac{L_t}{L_{t+1}} = \frac{1}{1+n}$ . This gives the law of motion for the capital-labor ratio:

$$k_{t+1} = \frac{1}{1+n} ((1 - \delta)k_t + sf(k_t)).$$

- Note that, for  $k_{t+1} \approx k_t$ , we can rewrite the above approximately as

$$k_{t+1} = (1 - \delta - n)k_t + sf(k_t).$$

- Higher population growth drags down the capital-labor ratio. And having higher  $n$  is *as if* having a higher capital depreciation. Why?

# Equilibrium in the Solow model

We can now summarize the equilibrium in the Solow growth model:

- Given initial  $k_0 = K_0/L_0$ , we can roll forward to solve  $\{k_t\}_{t=0}^{\infty}$ . And given  $\{k_t\}_{t=0}^{\infty}$ , we also find  $\{y_t\}_{t=0}^{\infty}$  using  $y_t = f(k_t)$ .
- An equilibrium of the Solow growth model refers to the whole path  $\{k_t, y_t\}_{t=0}^{\infty}$ .
  - Note: We also know the path of  $L_t$ . So we can back out the aggregate variables, for instance,  $K_t = k_t L_t$  and  $Y_t = y_t L_t$ , if necessary.
- We next turn to say more about properties of this equilibrium ...

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# Solving the equilibrium in the Solow model

- For now, suppose there is no population growth, i.e., that  $n = 0$ . The problem set deals with the case  $n \neq 0$ .
- Without population growth, the law of motion of the capital stock becomes

$$k_{t+1} = G(k_t) \equiv (1 - \delta)k_t + sf(k_t)$$

- We have reduced the equilibrium of the economy to a *first-order difference equation* in the economy's capital-labor ratio.
- This is something that we will see again and again: representing the macroeconomic equilibrium as a dynamic system, i.e., a set of difference or differential equations.

# Equilibrium characterization

## Proposition

*Given any initial point  $k_0 > 0$ , the economy's equilibrium dynamics are given by the path  $\{k_t\}_{t=0}^{\infty}$  such that*

$$k_{t+1} = G(k_t), \quad (1)$$

*for all  $t \geq 0$ , where*

$$G(k) \equiv sf(k) + (1 - \delta)k.$$

Equivalently, the growth rate of capital is given by

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = \gamma(k_t), \quad (2)$$

where

$$\gamma(k) \equiv s\phi(k) - \delta, \quad \phi(k) \equiv f(k)/k.$$

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# Steady state: definition

## Definition

The economy's “**steady state**” is a fixed point for the dynamics of  $k_t$ . That is,  $k^*$  is a steady state if and only if

$$k^* = G(k^*)$$

- Since  $G(0) = sf(0) = 0$ , there is a trivial steady state at  $k = 0$ .
- We henceforth focus on the non-trivial case in which  $k = k^* > 0$ .

# Steady state: existence and uniqueness

## Proposition

*Suppose  $\delta, s \in (0, 1)$  and  $f$  is a neoclassical production function. A steady state with  $k^* > 0$  exists and is unique.*

- By its definition,  $k^*$  solves  $k^* = G(k^*)$ , or

$$sf(k^*) = \delta k^*.$$

- Equivalently,

$$\phi(k^*) = \frac{\delta}{s} \tag{3}$$

where the function  $\phi$  gives the average product of capital:

$$\phi(k) \equiv \frac{f(k)}{k}.$$

- We infer that characterizing the steady state of the economy reduces to the simple task of characterizing solutions to equation (3).

## Steady state: existence and uniqueness

- The properties of  $f$  imply that  $\phi$  is continuous (and twice differentiable), decreasing, and satisfies the Inada conditions at  $k = 0$  and  $k = \infty$ :

$$\phi'(k) = \frac{f'(k)k - f(k)}{k^2} = -\frac{F_L}{k^2} < 0,$$
$$\phi(0) = f'(0) = \infty \quad \text{and} \quad \phi(\infty) = f'(\infty) = 0,$$

where the last two properties follow from L'Hôpital's rule.

Summary: the LHS  $f'$ n starts at  $\infty$ , is continuous, slopes downward, and goes to 0.

- Thus, if  $\delta > 0$  and  $s > 0$  (which we have assumed), then (3) has a solution. The monotonicity of  $\phi$  then guarantees that the solution is unique.
- We conclude that a steady-state level of capital exists, is unique, and is given by

$$k^* = \phi^{-1} \left( \frac{\delta}{s} \right),$$

which completes the proof. □

# Steady state: illustration

- We can rewrite the law of motion as

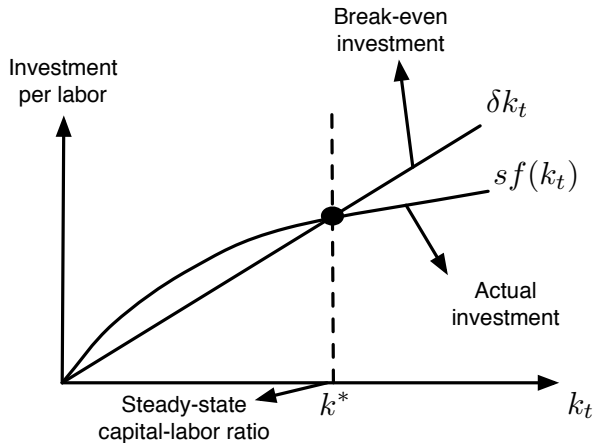
$$k_{t+1} - k_t = sf(k_t) - \delta k_t.$$

- Recall that  $sf(k_t)$  is actual investment (per labor).
- Let us refer to  $\delta k_t$  as the break-even investment.  
i.e., investment necessary to replenish the depreciated capital.
- We then have that a steady-state  $k^*$  is any (positive) solution to

$$\underbrace{sf(k^*)}_{\text{actual investment}} = \underbrace{\delta k^*}_{\text{break-even investment}} \quad (4)$$

- We can visualize these objects on a diagram ...

# Steady state: illustration



Visual “proof”:  $sf(0) = s \cdot 0$ ,  $sf'(0) = \infty$ ,  $sf'(\infty) = 0$ ,  $sf'(k) \geq 0$ , so must cross line with slope  $\delta$  at some point. Only once because of concavity.

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# Transitional dynamics

- The preceding analysis has characterized the (unique) steady state of the economy.
- Naturally, we'd like to know whether the economy will converge to the steady state if it starts away from it.
  - Another way to ask the same question is whether the economy will eventually return to the steady state if an exogenous shock perturbs the economy away from the steady state.
  - I.e., is the steady state stable? Is it a “resting point” of the dynamic system?
- The following proposition uses the properties of the functions  $G$  and  $\gamma$  (defined before) to establish that, in the Solow model, convergence to the steady state is always ensured and is indeed monotonic.

# Transitional dynamics

## Proposition

*Given any initial  $k_0 \in (0, \infty)$ , the economy converges asymptotically to the steady state:*

$$\lim_{t \rightarrow \infty} k_t = k^*$$

*Moreover, the transition is monotonic:*

$$k_0 < k^* \implies k_0 < k_1 < k_2 < k_3 < \dots < k^*$$

$$k_0 > k^* \implies k_0 > k_1 > k_2 > k_3 > \dots > k^*$$

*Finally, the growth rate  $\gamma_t$  is positive and decreases with  $t$  towards zero if  $k_0 < k^*$  whereas it is negative and increases with  $t$  towards zero if  $k_0 > k^*$ .*

# Proof of transitional dynamics

- Recall that  $k_{t+1} = G(k_t) = sf(k_t) + (1 - \delta)k_t$ .
- We know that  $G'(k) = sf'(k) + (1 - \delta) > 0$  and  $G''(k) = sf''(k) < 0$ , i.e.,  $G$  is strictly increasing and strictly concave. We also know that  $G(0) = 0$ ,  $G'(0) = \infty$ ,  $G(\infty) = \infty$ ,  $G'(\infty) = (1 - \delta) < 1$ . Finally, by definition of  $k^*$ , we know that  $G(k) = k$  iff  $k = k^*$ .
- From these properties we can conclude the following:

- Suppose that  $k_t < k^*$ . Then  $k_{t+1} = G(k_t) > k_t$ . Why? We have

$$\frac{k_{t+1} - k_t}{k_t} = s \frac{f(k_t)}{k_t} - \delta > s \frac{f(k^*)}{k^*} - \delta = 0$$

since  $f(k)/k$  is decreasing in  $k$ . Thus  $k_t < k_{t+1}$  whenever  $k_t \in (0, k^*)$ .

- We also have

$$k_{t+1} - k^* = G(k_t) - G(k^*) = - \int_{k_t}^{k^*} G'(k) dk < 0$$

so if  $k_t < k^*$  then also  $k_{t+1} < k^*$ .

# Proof of transitional dynamics

- Putting everything together, by monotonicity,  $k_t$  converges asymptotically to some  $\hat{k} \leq k^*$ . By continuity of  $G$ ,  $\hat{k}$  must satisfy  $\hat{k} = G(\hat{k})$ , that is  $\hat{k}$  must be a fixed point of  $G$ . But we already proved that  $G$  has a unique fixed point, which proves that  $\hat{k} = k^*$ .
- A symmetric argument proves that, when  $k_0 > k^*$ ,  $\{k_t\}_{t=0}^{\infty}$  is strictly decreasing and again converges asymptotically to  $k^*$ .

# Proof of transitional dynamics

- Next, consider the growth rate of the capital stock. This is given by

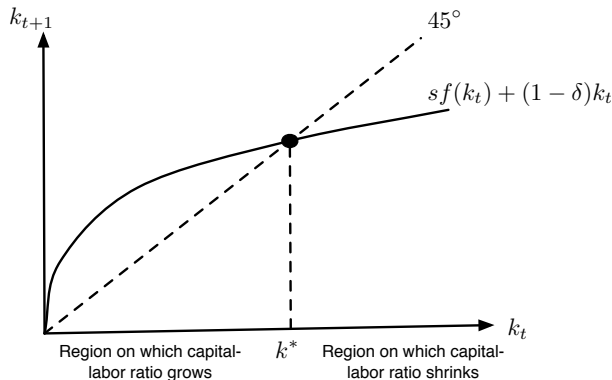
$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = s\phi(k_t) - \delta \equiv \gamma(k_t).$$

- Note that  $\gamma(k) = 0$  iff  $k = k^*$ ,  $\gamma(k) > 0$  iff  $k < k^*$ , and  $\gamma(k) < 0$  iff  $k > k^*$ . Moreover, by diminishing returns,  $\gamma'(k) = s\phi'(k) < 0$ .
- It follows that  $\gamma(k_t) > \gamma(k_{t+1}) > \gamma(k^*) = 0$  whenever  $k_t \in (0, k^*)$  and  $\gamma(k_t) < \gamma(k_{t+1}) < \gamma(k^*) = 0$  whenever  $k_t \in (k^*, \infty)$ .
- This proves that  $\gamma_t$  is positive and decreases towards zero if  $k_0 < k^*$  and it is negative and increases towards zero if  $k_0 > k^*$ .



# Illustration of the transitional dynamics

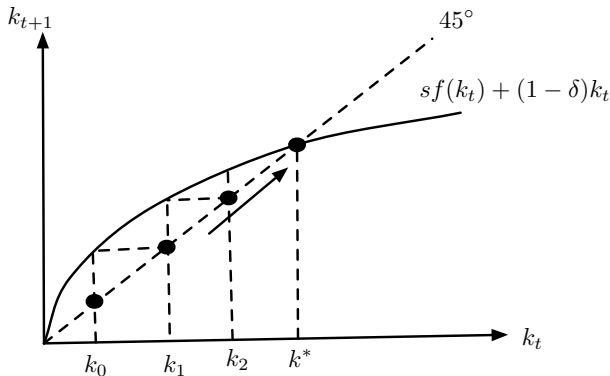
- To visualize the transition, we plot the function  $G$ , i.e.,  $k_{t+1} = (1 - \delta)k_t + sf(k_t)$ .



- $G$  inherits the curvature of  $f$ :  $G$  is concave because of diminishing returns to capital.
- It satisfies  $G(0) = 0$ ,  $G'(0) = \infty$ , and  $G'(\infty) = 1 - \delta < 1$ , again by the properties of  $f$ .

# Illustration of the transitional dynamics

- We can now represent the equilibrium path of the economy starting from any given  $k_0$ :



- What happens to  $y_t = f(k_t)$  along the equilibrium path?
- What happens to the size of each step in this path as we move closer to the steady state?
- What happens to  $k_t$  and  $y_t$  if we have  $k_0$  higher than  $k^*$ ?

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# Summary of the equilibrium characterization

- Without population growth ( $n = 0$ ), the equilibrium path is the solution to the following first-order difference equation:

$$k_{t+1} = (1 - \delta)k_t + sf(k_t)$$

with an initial condition  $k_0 = \bar{k}_0 > 0$ .

- The steady-state capital  $k^* > 0$  is found by solving

$$sf(k^*) = \delta k^*, \text{ which also gives } y^* = f(k^*).$$

- If  $k_0 = k^*$ , capital-labor ratio stays there,  $k_t = k^*$  for each  $t$ .
- If  $k_0 \neq k^*$ , capital-labor ratio converges monotonically to  $k^*$ .
  - Growth rate of  $k$  converges monotonically to zero (from the positive side if  $k_0 < k^*$ , and from the negative side if  $k_0 > k^*$ )
  - Behavior away from  $k^*$  is known as transitional dynamics.

# Summary of the equilibrium characterization

- We have therefore characterized the equilibrium.
- Without numerically specifying  $f$ ,  $s$  and  $\delta$ , we can't do much more calculation.
- But even in this general form, the characterization has **testable implications**, which we turn to next ...

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# Testable implications of the Solow model

We'll now derive (and later test) **three key implications** of the Solow growth model:

1. Capital accumulation can't drive long-run growth.
2. Similar countries at different stages of development (different  $k_0$ ) will necessarily converge over time
3. Non-convergence requires either:
  - a) Modifications of the baseline Solow model: break convergence to a unique steady state
  - b) Differences in fundamentals  
I.e., differences that deliver different  $k^*$ , e.g., different production function  $f(\bullet)$  or saving rate  $s$ .

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# Capital accumulation can't drive long-run growth

- Consider three possibilities,  $k_0 < k^*$ ,  $k_0 = k^*$  or  $k_0 > k^*$ ?
- In which case is the economy growing? Is it growing perpetually, as in the data, or is growth coming to an end? Why?

# Capital accumulation can't drive long-run growth

- The Solow model was developed to understand growth.
- It can generate **temporary growth** during transitional dynamics starting with low  $k_0$ , but produces **no long-run growth**!
- Intuitively, diminishing returns to capital will eventually restrain growth. Capital accumulation becomes less and less useful.
- So need to change the model (later) to fight diminishing returns.
  - This is how even predictions that fail can be very useful.
  - Moreover, the model's other predictions have more truth ...

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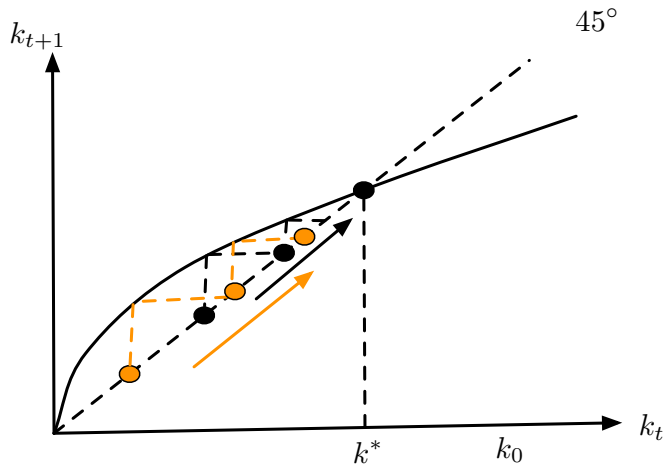
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# Conditional convergence



# Conditional convergence

- **Assumption:**

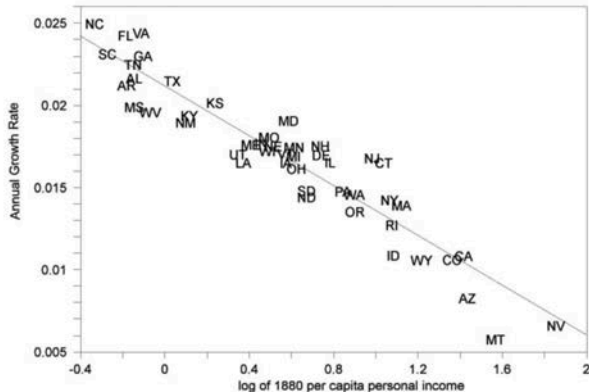
- Suppose two countries have different levels of economic development (namely different initial levels  $k_0$  and thus  $y_0$ ) ...
- ... but otherwise share the same fundamental characteristics (namely share the same technologies, saving rates, depreciation rates, and fertility rates).

- **Implication: conditional convergence.** Then the poorer country will grow faster than the richer one and will eventually catch up with it.

- Can you imagine economies that share similar “fundamentals” (institutions, policies, know how) and hence similar steady states but may have started from different initial conditions?
- Did those economies grow faster in the data?

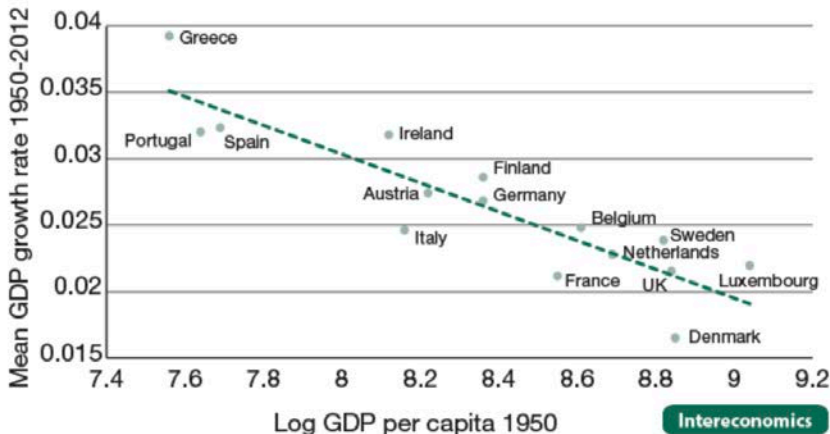
# Convergence across the US

Figure 4: Convergence of Personal Income across U.S. States.  
1880 Personal Income and Income Growth from 1880 to 1990



Strong support for conditional convergence across US states.

# Convergence across Europe



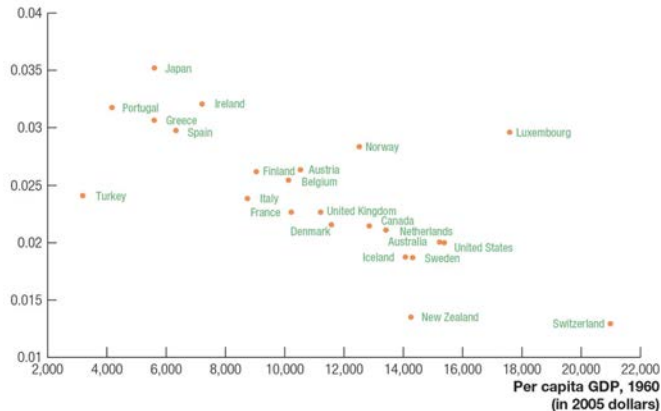
... similar for European Union member states

# Convergence across OECD countries

FIGURE 5.8

## Growth Rates in the OECD, 1960–2010

Per capita GDP growth  
(1960–2010)

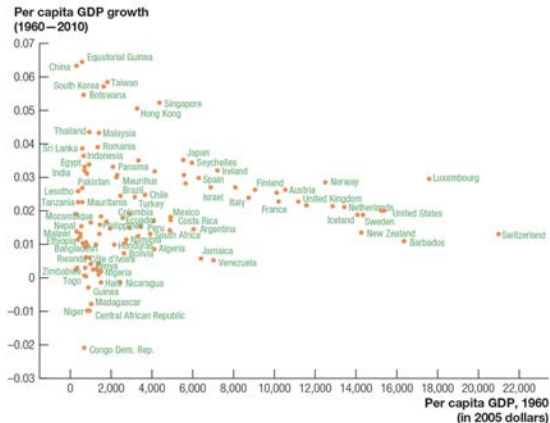


... and also for OECD countries

# Divergence for a broader set of countries

FIGURE 5.9

Growth Rates around the World, 1960–2010



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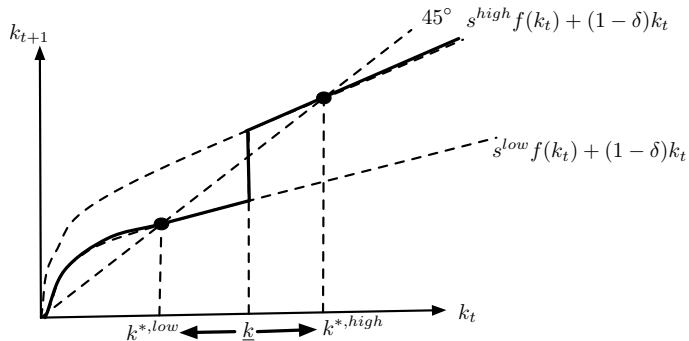
# Divergence

- Can we make sense of persistent long-run differences *within* the Solow growth model?
- Two main options:
  - a) Introduce non-linearities to allow for multiple steady states (so-called “poverty traps”)
  - b) Permanent heterogeneity across countries in fundamentals ( $f, s, \delta$ )

# Divergence with poverty traps

- To illustrate the basic idea, suppose the savings behavior is more complicated than what we assumed so far.
- If  $k$  is lower than some  $\underline{k}$ , then the savings rate is relatively low,  $s^{low}$ .
  - Imagine consuming for subsistence—not much room for savings when  $k, y$  are low.
- If  $k$  exceeds  $\underline{k}$ , then the savings rate increases to  $s^{high} > s^{low}$ .
- Under these assumptions, we could have interesting dynamics ...

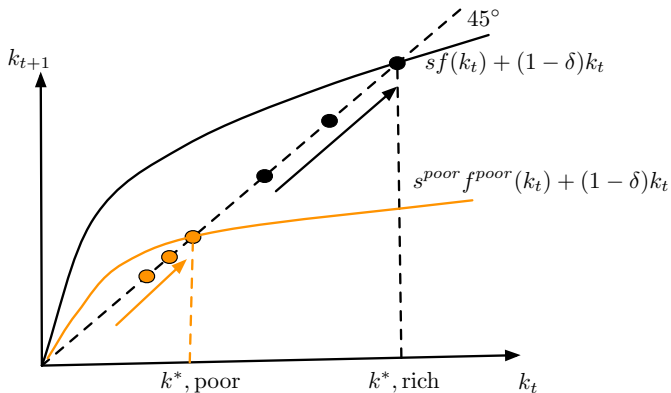
# Multiple steady states and poverty traps



# Multiple steady states and poverty traps

- In this case, there are multiple steady-states,  $k^{*,low}$  and  $k^{*,high}$ .
  - If country starts close to  $k^{*,low}$ , converges to  $k^{*,low}$  and stays there.
  - But if the country were to somehow get close to  $k^{*,high}$ , then it would converge to  $k^{*,high}$  and instead stay at that higher level.
- So the low steady-state  $k^{*,low}$  represents a so-called **poverty trap**.

# Divergence in the basic Solow model



# Multiple steady states and poverty traps

- Key implication of this theory: poor countries are poor because they are stuck.
- Used to motivate international aid and “big push” type ideas—help economies to get close to  $k^{*,high}$ .
- But also raises the question of whether the equilibrium level of savings, and any associated poverty trap, are inefficient or not.
  - E.g., what if getting the next generation out of a poverty trap requires starving—or at least imposing material hardship on—the current generation?
- We will revisit these ideas later.

# Permanent heterogeneity

- Now instead assume differences in steady-states,  $k^*, y^*$ .
  - Ultimately driven by differences in characteristics (different  $f(\cdot)$ ,  $s$  etc).
  - If  $k^{*,j} < k^{*,US}$ , then country  $j$  will not converge to the US. Why?
- So differences in  $k^*$  can at least explain (some of) the cross country income differences. But what determines if a country has a high or low  $k^*$ ?
- We address this by considering what's referred to as **comparative statics** for  $k^*$  ...

# Comparative statics (& dynamics)

- Comparative statics will be our way of doing (thought) experiments.
  - **Comparative statics:** How a change in parameters (e.g., savings rate, productivity) affects the steady state  $(k^*, y^*)$ .
  - **Comparative dynamics:** How a change affects the whole equilibrium path,  $\{k_t, y_t\}_{t=0}^{\infty}$ . Including transitional dynamics from  $k_0 = k^{*,old}$  to  $k^{*,new}$ .
- We will mostly study comparative statics, but also some graphical explorations for comparative dynamics. More in problem set.

# Comparative statics of $k^*$ and $y^*$

- Let's bring back the TFP coefficient  $A$  to the production function, so that  $y = Af(k)$ .
- Then the steady-state is characterized by the equation:

$$sAf(k^*) = \delta k^*.$$

- The steady-state levels of capital per worker,  $k^*$ , and output per worker,  $y^*$ , are ...
  - ... increasing in the savings rate,  $s$ , and the productivity,  $A$
  - ... decreasing in the depreciation rate,  $\delta$
- Make sure you understand how to show this using the graphical analysis we developed earlier. How about a formal argument? Implicit function theorem?

# A productivity-increasing reform

- Now consider a country that is initially at its steady state,  $k_0 = k^{*,old}$ .
- Suppose the country undertakes a reform that doubles its productivity from some initial level  $A^{old}$  to  $A^{new} = 2A^{old}$ .
  - How does this affect output in the short run,  $y_0^{new}$  vs  $y_0^{old}$ ?
  - How does the reform affect output in the long run after the country converges to its new steady-state,  $y^{*,new}$  vs  $y^{*,old}$ ?

# A productivity-increasing reform

- In the short run, the capital stock,  $k_0$ , is fixed. We compare

$$y_0^{new} = A^{new} f(k_0) \quad \text{vs.} \quad y_0^{old} = A^{old} f(k_0).$$

We have  $y_0^{new}/y_0^{old} = A^{new}/A^{old} = 2$ . Output also doubles.

- In the long run, the country converges to  $k^{*,new}$ . So we compare:

$$y^{*,new} = A^{new} f(k^{*,new}) \quad \text{vs.} \quad y^{*,old} = A^{old} f(k^{*,old})$$

- From the earlier result, we know that  $k^{*,new} > k^{*,old}$ . Why?
  - So  $y^{*,new}/y^{*,old} > A^{new}/A^{old} = 2$ . Output more than doubles!
- Intuitively, greater productivity induces greater investment, which ultimately increases productive capacity even further.

## A productivity-increasing reform

- The above results hold for *any* neoclassical production function. It might be useful to illustrate them for the Cobb-Douglas case,  $y = Ak^\alpha$ .
- In this case, we can explicitly solve for the steady-state as (do it):

$$k^* = \left( \frac{sA}{\delta} \right)^{\frac{1}{1-\alpha}},$$
$$y^* = A \left( \frac{sA}{\delta} \right)^{\frac{\alpha}{1-\alpha}} = A^{\frac{1}{1-\alpha}} \left( \frac{S}{\delta} \right)^{\frac{\alpha}{1-\alpha}}.$$

- **Comparative statics:**  $k^*$  is increasing in  $s, A$ , decreasing in  $\delta$ .
  - **Amplification:**  $A$  has exponent  $1/(1-\alpha) > 1$  in  $y^*$  equation.
- The larger the income share of capital  $\alpha$  (equivalently, the less powerful the diminishing returns to capital), the stronger the effects of  $s$  and  $A$  on  $y^*$ . (Intuition?)

## Empirical test: savings vs. $k/y$

- Let's use this theory for a simple first test.
- Recall we have the equation,  $sy^* = \delta k^*$  (where  $y^* = Af(k^*)$ ). This suggests considering the  $k/y$  ratio (instead of  $k$  or  $y$ ):

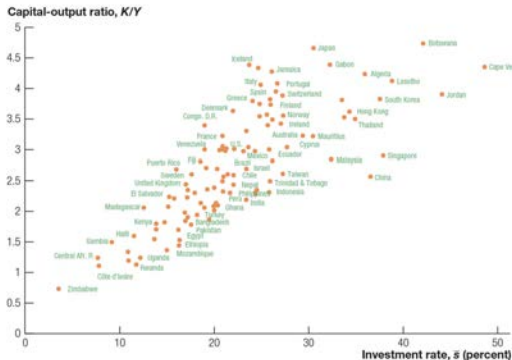
$$\frac{k^*}{y^*} = \frac{s}{\delta}.$$

- The depreciation rate  $\delta$  doesn't vary much across countries.
  - So countries with higher savings rate  $s$  have greater (steady state) capital relative to output (regardless of  $A$  or  $f(\cdot)$ ).
  - Intuitively, saving a greater fraction of output should eventually lead to a greater stock of capital relative to output.
- We can test this prediction, once we measure  $y$  and  $k$ ...

# Empirical test: savings vs. $k/y$

FIGURE 5.3

## Explaining Capital in the Solow Model



- The broad patterns we see are very much consistent with model.
- But: while in the Solow model causality runs one way only, from the saving rate to  $K/Y$  ratio, in the real world causality could run in the opposite direction as well.

# Taking Stock

# Taking stock from the Solow model

- We have seen the Solow model with its key implications:
  1. Capital accumulation is not the engine of long-run growth.
  2. Conditional convergence (conditional on steady-state).
  3. Cross country differences in steady-states.  
Differences in  $s, A$  can explain some differences in  $k^*, y^*, k^*/y^*$ .
- We have also seen some key concepts: equilibrium, steady state, transitional dynamics, comparative statics/dynamics ...
- Next up: analyze the model's *cross country* predictions quantitatively—where do  $y^*$  differences come from?  $s$  or  $A$ ?

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