

# **Lecture 7: Growth Accounting (continued), Golden Rule, and Endogenous Saving**

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14.05, Fall 2025

# Today's roadmap

## 1. Additional applications of growth accounting.

Same idea as last time: show that this is a useful tool & prepare you for the project.

## 2. So far we've treated the savings decision as a blackbox. Now let's endogenize it. How to think about "optimal" saving?

- Will be proceeding in two steps:
  - (i) Solow model: golden rule, dynamic inefficiency.
  - (ii) Neoclassical growth model (NGM) = Ramsey model: optimizing households decide how much to save
- Aside: why focus on savings and thus  $k$  accumulation decision if we saw that  $A$  is key?
  - Will develop tools that are useful to think about endogenous  $A$ . In particular: we will later think about  $A$  also coming out of optimal "investment" decisions
  - Still important to figure out how much a country "should" save. Also: savings/investment are very important for business cycles (second half).

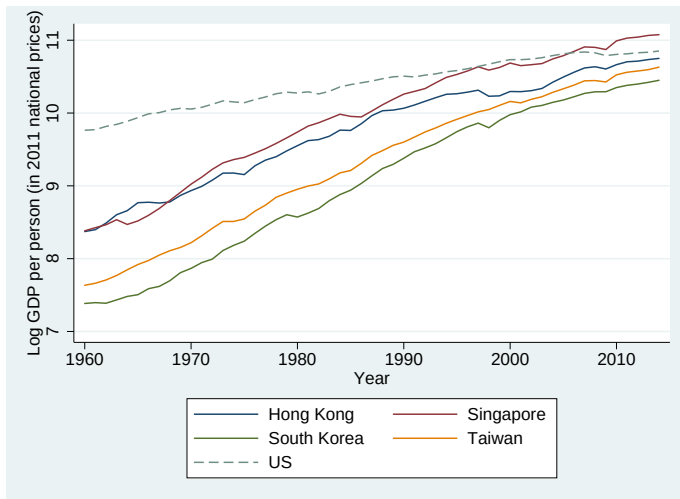
# Outline

1. Additional applications of growth accounting
2. Towards the Neoclassical Growth Model
  - Warm-Up: Golden rule and dynamic inefficiency
  - The optimal growth problem
  - Warm-up: 2-period model

# Growth Accounting

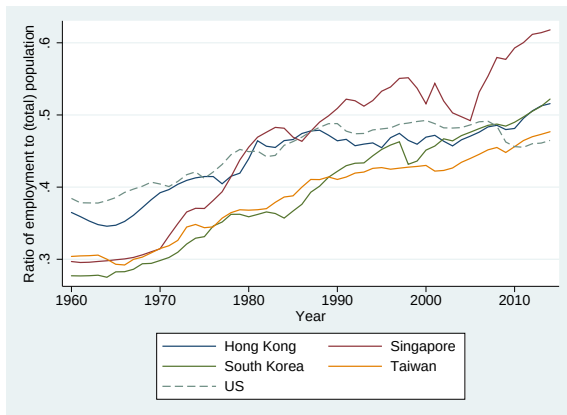
- Consider two further applications:
  1. Asian tigers
  2. China and India
- Same diagnostic tools, but different diagnoses/stories

# Four Asian tigers: output per person



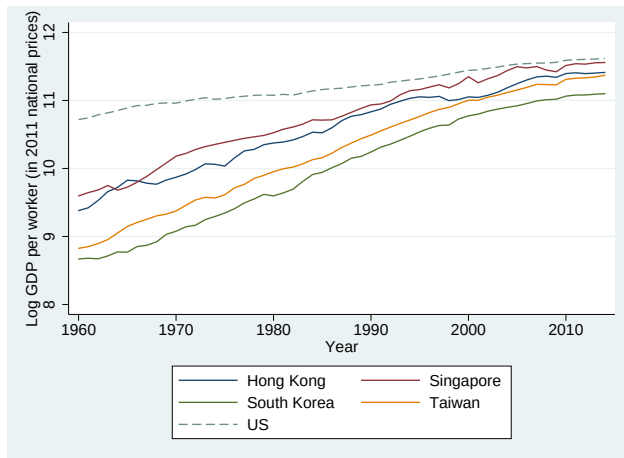
How did they grow so fast?

# Four Asian tigers: labor market participation



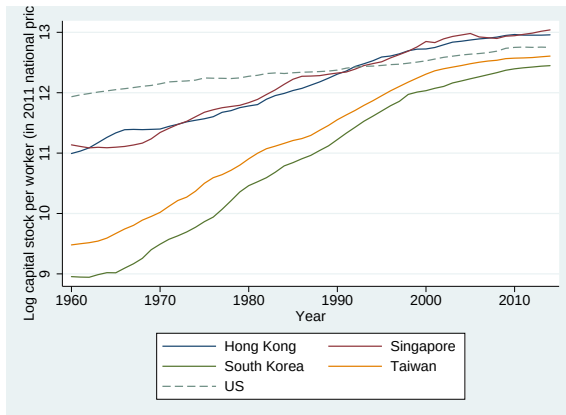
- Labor participation has been increasing in most of those tiger countries.
- What does this imply for the growth of output **per worker**,  $Y/L$ , compared to the growth of output per person?

# Four Asian tigers: output per worker



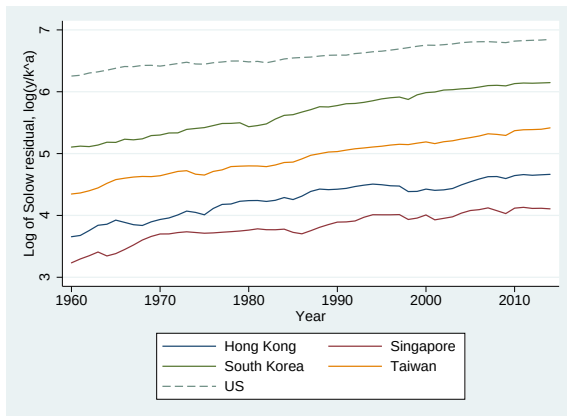
- Growth is less dramatic in terms of output per worker,  $Y/L$ , but still impressive—and higher than the US counterpart.

# Four Asian tigers: capital per worker



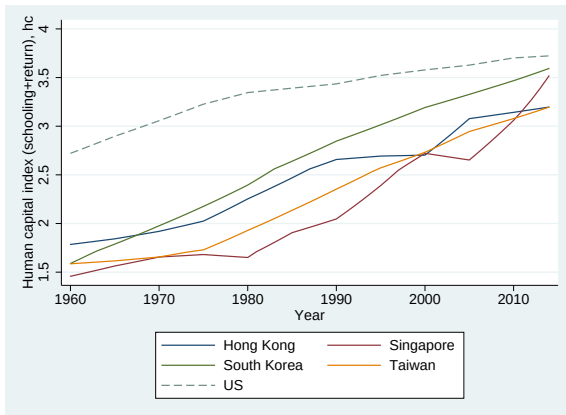
- $K/L$  has also been growing fast—even faster than  $Y/L$  (the baseline we expect if capital was simply responding to TFP growth).
- What does it mean for these countries' Solow residuals?

# Four Asian tigers: Solow residual or TFP



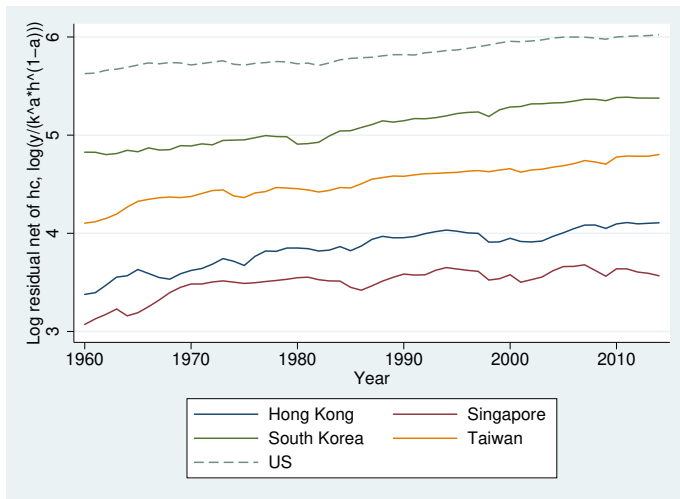
- Ignore human capital for now, measure TFP by simple Solow residual.
- Conclusion: although TFP levels are smaller than the US counterpart, TFP growth rates do not seem too different than in the US.

# Four Asian tigers: how about human capital?



- Rapid growth in human capital (driven by increase in schooling).
- What happens to the residual growth after we account for  $h$ ?

# Four Asian tigers: TFP, net of schooling

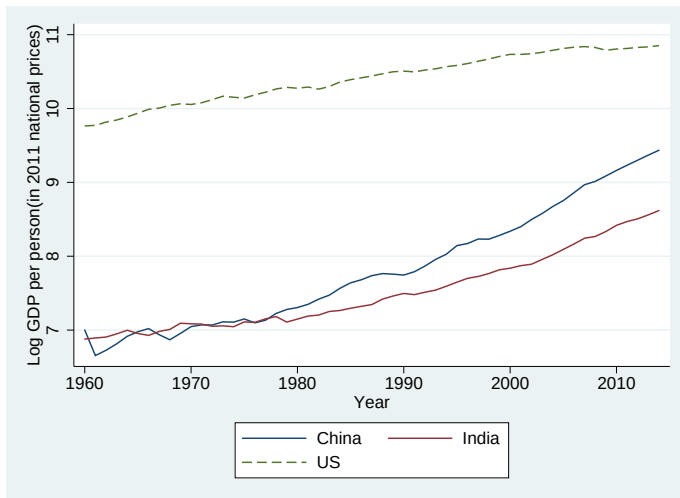


- Once we account for  $h$ , the TFP growth is even less impressive.

# Lessons from the Four Asian Tigers

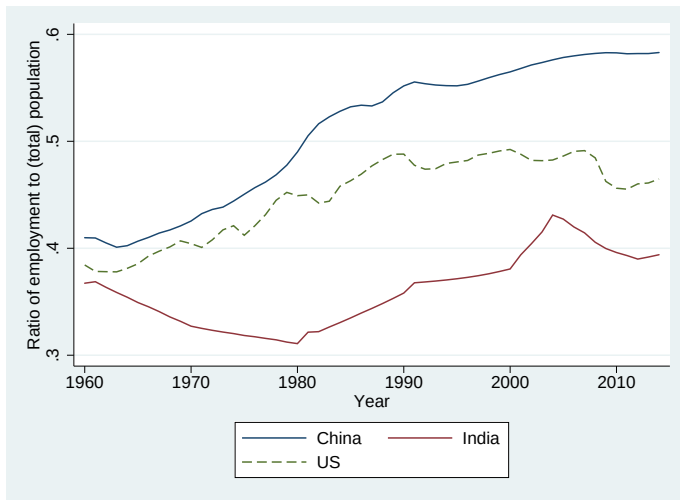
- Young (1995) does detailed growth accounting for these countries and confirms the results from our visual “growth accounting” .
- In all cases, **factor accumulation**—labor growth, schooling, and capital accumulation (except Hong Kong)—play an important role for growth.
- TFP plays a more minor role than what a naïve observer could think based on their exceptional growth rates.
- So need to focus on how they managed to **accumulate factors**.
  - Policies conducive to employment, savings, and education.
  - So for these countries our theories so far—which are all about the accumulation of factors of production—actually seem quite important
- Do these stories work with China and India? ...

# China and India: output per person

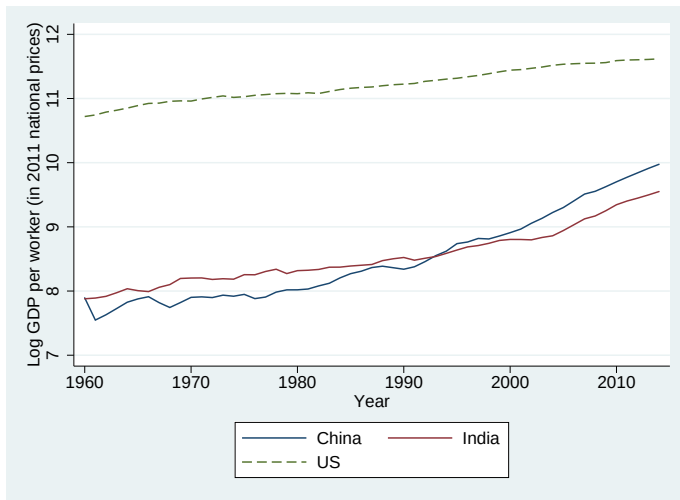


China and India: how did they grow so fast?

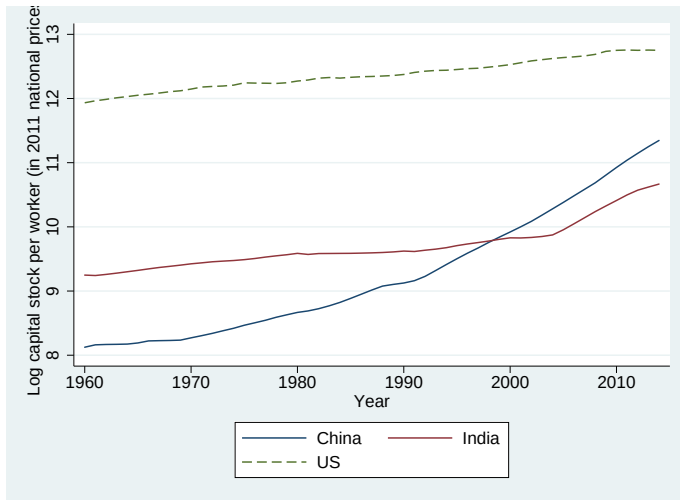
# China and India: labor market participation



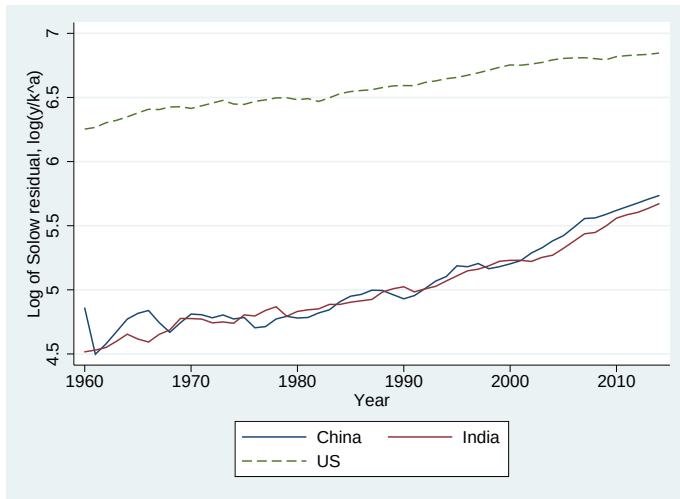
# China and India: output per worker



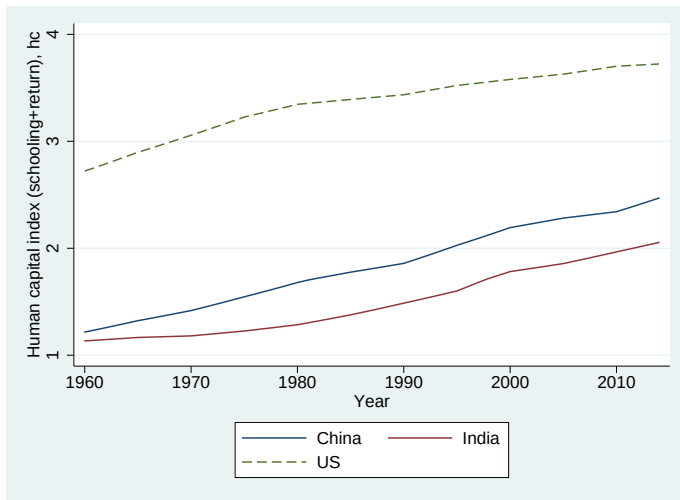
# China and India: capital per worker



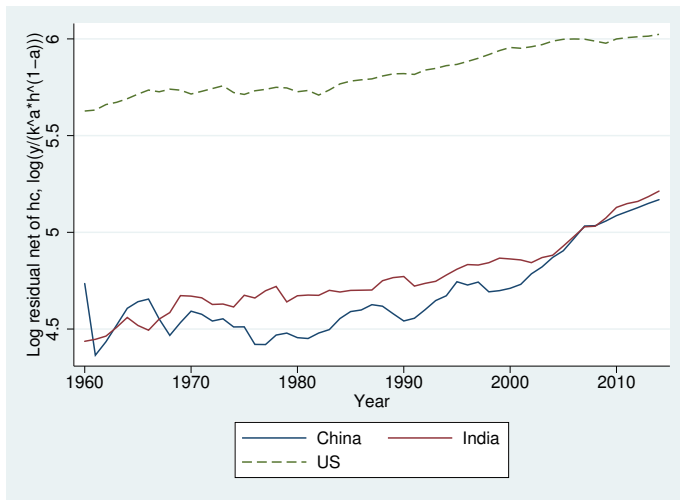
# China and India: Solow residual or TFP



# China and India: human capital



# China and India: TFP net of human capital



# Lessons from China and India

- For more detailed accounting, see Bosworth & Collins (2008), “Accounting for Growth: Comparing China and India,” *Journal of Economic Perspectives*.
- Bottom line: growth in China and India driven largely by TFP. Need to focus on how and why those countries managed to raise TFP.
  - Sharp contrast with the Four Asian Tigers.
  - Focus on policies conducive to raising productivity.
- If you want to dig further, read Zhu (2012), “Understanding China’s Growth: Past, Present, and Future,” *Journal of Economic Perspectives*.

# Outline

1. Additional applications of growth accounting

2. Towards the Neoclassical Growth Model

Warm-Up: Golden rule and dynamic inefficiency

The optimal growth problem

Warm-up: 2-period model

# From Solow model to NGM and beyond

- The Solow growth model and growth accounting are useful diagnostic tools. But they are a bit mechanical—no trade-offs, no optimal decisions.
  - Why did most of the tigers rapidly accumulate capital?
  - How did China and India grow their TFP in recent years?
- We need finer analysis and richer models. In particular need to relate key parameters, like  $s$  and  $g$ , to **incentives** and thus to **preferences**, **technology**, and **policies**.
- Our main tool to analyze these issues will be the **neoclassical growth model** (NGM) and its various extensions.
  - In the baseline NGM model, in particular, we will focus on the following question: how do countries trade off consumption and savings?
- As a preliminary step, let's revisit consumption in the Solow model...

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# How about consumption in the Solow model?

- So far, we used the Solow model to analyze output and capital growth. But ultimately individuals' welfare depends on consumption.
- Consider the Solow model with  $n = 0$  and  $g = 0$  (discrete time). Recall consumption (per worker) is the residual of savings:

$$c_t = \underbrace{f(k_t)}_{\text{output}} - \underbrace{sf(k_t)}_{\text{investment}} = (1 - s)f(k_t).$$

- Let us characterize consumption in the steady state and investigate how it depends on the country's savings rate ...

# Steady-state consumption in the Solow model

- Recall the economy converges to a unique steady state, where that steady state has

$$sf(k^*) = \delta k^* \quad (1)$$

- Think of this as determining capital as a function of savings,  $k^*(s)$ .
  - Note that this is unambiguously an increasing function: a higher saving rate translates to a higher capital-labor ratio in steady state. Thus output is also increasing in  $s$ .
- At the steady state, we can write consumption as a function of savings:

$$c^*(s) = (1 - s)f(k^*(s))$$

- What happens to  $c^*$  if  $s = 0$ ? What is the intuition?
  - What happens to  $c^*$  if  $s = 1$ ? What is the intuition?
- So neither extreme looks very desirable for raising  $c^*$ .

# Maximizing the steady-state consumption

- Suppose (for now) that our goal is simply to maximize  $c^*$ . What is the savings rate  $s \in [0, 1]$  that achieves this?
- Recall that, using (1), we can write:

$$\begin{aligned}c^*(s) &= f(k^*(s)) - sf(k^*(s)) \\ &= f(k^*(s)) - \delta k^*(s),\end{aligned}$$

- So we want to solve the optimization problem:

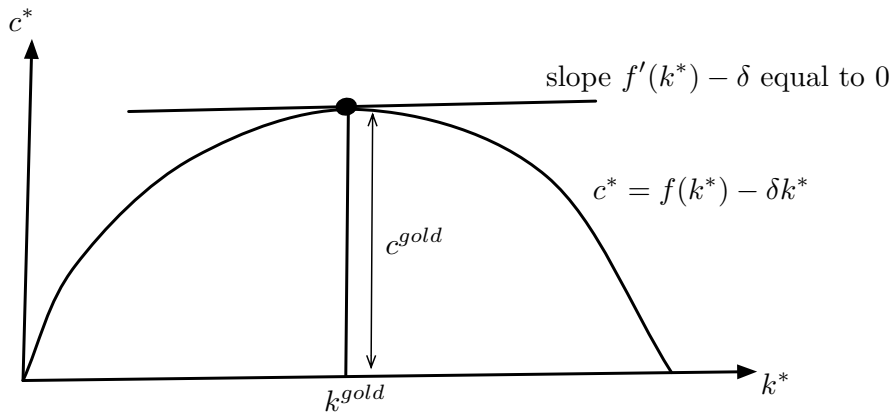
$$\max_{s \in [0,1]} c^*(s) = f(k^*(s)) - \delta k^*(s)$$

This is equivalent to solving the problem:

$$\max_{k^* \in [0, k^*(1)]} c^* = f(k^*) - \delta k^*. \quad (2)$$

- We can then back out the implied  $s \in [0, 1]$  using  $k^*(s)$ .

## Solution: golden rule capital labor ratio



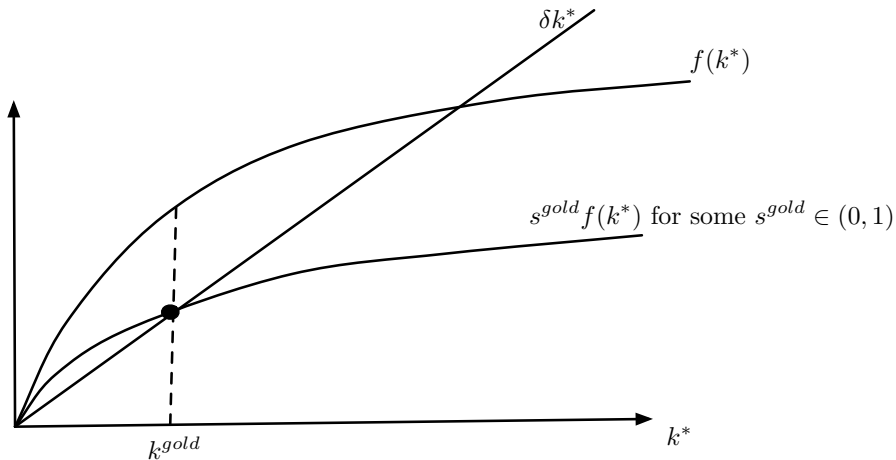
## Solution: golden rule capital labor ratio

- The solution to problem (2) is given by  $k^{gold}$  such that

$$f'(k^{gold}) = \delta$$

- The level,  $k^{gold}$ , is known as the “golden rule” level. Intuition?
  - At this level, the benefit of more steady-state capital (more production) is exactly balanced by the cost (more replacement investment).
  - The figure on the next slide will provide a visual illustration
- How do we find the saving rate,  $s^{gold} \in [0, 1]$ , that leads to  $k^{gold}$ ?

# Golden rule



Next **Q**: what saving rate *should* an economy pick? is it obvious that it should be  $s^{gold}$ ?

# Dynamic inefficiency

What happens if the saving rate happens to be above  $s^{gold}$ ?

- To fix ideas, suppose the economy saves at rate  $s^{\max} = 1$ . Consumption is then:

$$c^* = f(k^{\max}) - \delta k^{\max} = 0.$$

- Suppose the economy starts with  $k_0 = k^{\max}$ . But suddenly people start saving at a new rate  $s < s^{\max} = 1$ .
- Would consumption increase/decrease/remains unchanged in the long run (steady state)? How about in the short run (e.g., period 0)? Hint: Recall that  $c_t = (1 - s)f(k_t)$  for all  $t$ .
- Would “reasonable” people choose the rate  $s^{\max} = 1$ ?
- In general: consumption can be increased **in all periods** if  $s > s^{gold}$ .

Why? consumption jumps up at date 0 and then monotonically decreases towards  $c^{gold} > c^*(s)$ , so  $c_t > c^*(s)$  for all  $t$ . Let's complete the argument together.

- Known as dynamic inefficiency. So suboptimal outcomes can happen in the Solow model.

# The optimal savings rate

- So  $s > s^{gold}$  is dominated in terms of attainable consumption. What about  $s < s^{gold}$ ?
  - $s < s^{gold}$  is not similarly dominated. Why?
  - Dynamic trade-off: increasing the savings rate lowers consumption today but increases it in the future. Unclear what to do.
- So what savings rate should we pick?
  - Golden rule seems sensible from a very long-run perspective. But how should we evaluate the transition? Is it ok to hurt current generations?
  - More generally, why should we force ourselves to pick a constant savings rate? Why not let it evolve and change over time?
- That's what we'll do next: pick savings rates *optimally*, in particular allowing them to change over time.

This is known as the neoclassical growth model. Relates savings  $s$  to **preferences/technology/policies**.

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# Notation and basics

- For simplicity, we consider the case with no growth,  $n = g = 0$ . We will work with per labor variables, as usual, denoted  $\{k_t, y_t\}_{t=0}^{\infty}$ .

As before, we can generalize to  $n, g > 0$  with similar insights.

- Also as before we assume that the production function  $f(k_t) = F(k_t, 1)$  is neoclassical.
- Differently from before, we will now pay particular attention to consumption (per labor), denoted  $\{c_t\}_{t=0}^{\infty}$ .

# Technology and the resource constraint

- Output is split between consumption and investment, as usual:

$$c_t + i_t = f(k_t)$$

- Recall that capital accumulates according to

$$k_{t+1} = (1 - \delta)k_t + i_t.$$

- Combining the above, we can rewrite **resource constraint** of the economy as

$$c_t + k_{t+1} = \underbrace{\tilde{f}(k_t) \equiv f(k_t) + (1 - \delta)k_t}_{\text{available resources}},$$

These are the technological constraints. They are exactly as in the Solow model.

- We define the function,  $\tilde{f}(\cdot)$ , to simplify the notation.

# Preferences for consumption

- What's left? Savings decision. Before we just assumed a constant, fixed savings rate.
- Now: savings are chosen optimally according to some preferences of how households value consumption, trading off the present vs. future.
- We assume that the economy's preferences are captured by the **utility function**

$$U_0 = \sum_{t=0}^{\infty} \left( \frac{1}{1+\rho} \right)^t u(c_t)$$

- $u(\cdot)$  is the per-period utility function. Increasing and strictly concave in  $c_t$ . Captures how much households like additional units of consumption.
- The term  $\beta \equiv \frac{1}{1+\rho} \in (0, 1)$  is called the discount factor. The parameter  $\rho > 0$  is called the discount rate. Captures preference of current vs. future consumption.

We're getting back to something like the RC economy: now have a trade-off, here between consumption today and tomorrow, back then between consumption and leisure.

## Example: CES utility

- We typically work with the following per-period utility function:

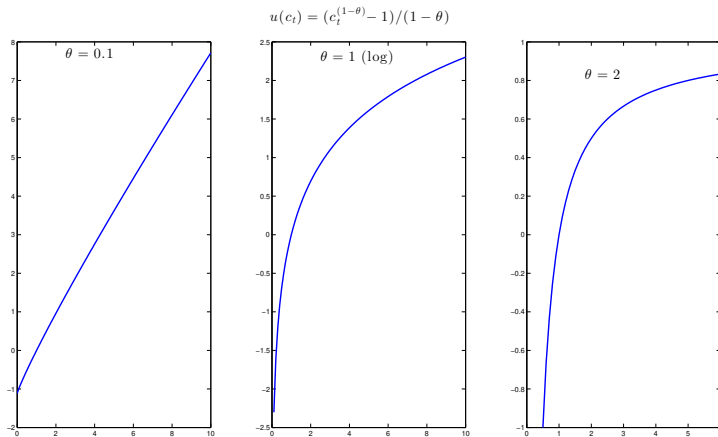
$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta} \text{ given } \theta > 0$$

- Note that the marginal utility of consumption is given by

$$u'(c) = c^{-\theta}$$

- So marginal utility from consumption is diminishing in  $c$ . When  $\theta$  is greater, marginal utility is diminishing faster.
  - As  $\theta \rightarrow 0$ , the function becomes linear  $u(c) = c$ .
  - As  $\theta \rightarrow 1$ , the function becomes  $u(c) = \log c$ .
  - For  $\theta > 1$ , the function becomes more concave than log.
- Known as **constant elasticity of substitution (CES)** utility.

# Illustration of the CES utility



$1/\theta$ , captures **the elasticity of substitution**.

Low  $\theta$ : More willing to substitute consumption across periods. This intuition will become clearer later.

# Optimal growth problem

- We have preferences and technology. What are our predictions for what will happen?
- Will again consider two modes of organizing production:

Just as in our Robinson Crusoe economy.

## 1. **Benevolent social planner** [we will start with this]

- Optimal growth problem: planner chooses consumption and savings to maximize the representative household's utility
- The solution to this problem will indicate how well the economy could possibly do, i.e., the so-called “optimal growth”/Ramsey problem

## 2. **Market economy** [this will come later]

- Households: supply labor and save in capital to maximize their utility, firms rent capital and labor, both take prices as given, look for market-clearing prices
- Will see: same result as planning solution, yet again “invisible hand”

# Optimal growth problem (Ramsey Problem)

- A social planner chooses a plan  $\{c_t, k_{t+1}\}_{t=0}^{\infty}$  so as to maximize utility subject to the resource constraint of the economy, taking initial  $k_0$  as given. Formally:

$$\max \sum_{t=0}^{\infty} \left( \frac{1}{1+\rho} \right)^t u(c_t)$$

$$\text{subject to: } c_t + k_{t+1} = \tilde{f}(k_t) = f(k_t) + (1-\delta)k_t,$$

$$k_0 > 0 \text{ given,}$$

$$k_{t+1} \geq 0 \text{ and } c_t \geq 0.$$

- In Lecture 3 we supported the “aggregate production function” with wages and interest rates. Soon we will see how to decentralize the intertemporal savings decision implied by this Ramsey growth problem via wages and interest rates.
- Challenging: this is an *infinite-horizon* optimization problem.

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# Optimal growth problem: two periods

- Suppose, to make our lives initially very easy, that the economy just exists for two periods 0 and 1.
- In this special case, the planner's problem looks as follows:

$$\begin{array}{ll}\max_{c_0, k_1, c_1, k_2 \geq 0} & u(c_0) + \frac{1}{1+\rho} u(c_1), \\ \text{s.t.} & c_0 + k_1 = \tilde{f}(k_0), \\ & c_1 + k_2 = \tilde{f}(k_1).\end{array}$$

# Solving the problem with two periods

- Given that there are only two periods, the planner finds optimal to choose

$$k_2 = 0$$

Do not leave behind any unconsumed capital. Why?

- We can then solve for consumption from the two resource constraints:

$$\begin{aligned}c_1 &= \tilde{f}(k_1), \\c_0 &= \tilde{f}(k_0) - k_1.\end{aligned}$$

## Solving the problem with two periods

- Using these, we can write planner's problem as:

$$\max_{k_1 \geq 0} \underbrace{u(\tilde{f}(k_0) - k_1)}_{\text{current consumption}} + \frac{1}{1 + \rho} \underbrace{u(\tilde{f}(k_1))}_{\text{future consumption}}$$

- So the planner's problem boils down to a single choice—of  $k_1$ . The interior optimum satisfies the first order condition:

$$u'(\tilde{f}(k_0) - k_1) = \frac{1}{1 + \rho} \tilde{f}'(k_1) u'(\tilde{f}(k_1))$$

- Substituting  $c_0$  and  $c_1$  back, we obtain the so-called **Euler equation** ...

# Key optimality condition: Euler equation

- The Euler equation relates marginal utility in subsequent periods:

$$\underbrace{u'(c_0)}_{\text{marginal utility at 0}} = \frac{1}{1 + \rho} \underbrace{\tilde{f}'(k_1)}_{\text{marginal return on investment}} \times \underbrace{u'(c_1)}_{\text{marginal utility at 1}}$$

- This equation plays a central role throughout. Intuition? Why is there a  $1/(1 + \rho)$  on the right hand side?

## Euler equation more generally

- Recall that  $\tilde{f}(k) = f(k) + (1 - \delta)k$  denotes available resources.
- Hence, in terms of the original function  $f(\cdot)$ , the condition is

$$\underbrace{u'(c_0)}_{\text{marginal utility from consumption}} = \underbrace{\frac{1}{1 + \rho} \overbrace{(f'(k_1) + 1 - \delta)}^{\text{marginal (gross) return on investment}} u'(c_1)}_{\text{marginal utility from investment}}$$

- Marginal return on investment includes  $f'(k_1)$  and nondepreciated capital.
- Overall: planner trades off consumption and investment optimally.

## Euler equation with the CES form

- Recall that CES utility is given by

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \quad \text{with marginal utility } u'(c) = c^{-\theta}$$

- The corresponding Euler equation is

$$c_0^{-\theta} = \frac{1}{1+\rho} (f'(k_1) + 1 - \delta) c_1^{-\theta}$$

- After rearranging terms, we obtain:

$$\frac{c_1}{c_0} = \left( \frac{1}{1+\rho} (f'(k_1) + 1 - \delta) \right)^{1/\theta}$$

# Interpreting the Euler equation

- Note that consumption growth is positive,  $c_1/c_0 > 1$ , when

$$\frac{1}{1+\rho} (f'(k_1) + 1 - \delta) > 1$$

which is equivalent to

$$\underbrace{f'(k_1) - \delta}_{\text{marginal net return on investment}} > \underbrace{\rho}_{\text{discount rate}}$$

- When LHS is larger, consume more in the future than today.
  - When LHS is smaller, consume more today than in the future.
  - When LHS=RHS, consumption is constant over time.
- Make sure you understand the intuition for these effects.

# Interpreting the Euler equation

- Let's go back to the Euler equation once more:

$$\frac{c_1}{c_0} = \left( \underbrace{\frac{1}{1+\rho} (f'(k_1) + 1 - \delta)}_{\text{captures growth incentives. let's call it } X} \right)^{1/\theta} = X^{1/\theta}$$

So  $1/\theta$  mediates how fast consumption responds to incentives.

- Suppose  $X > 1$  so that consumption grows:
  - Higher elasticity  $1/\theta \implies$  consumption grows faster.
  - Lower elasticity  $1/\theta \implies$  consumption grows, but more slowly.

# Summary of the two-period model

- We can now state all of the equations describing the solution to our model:
  - The key optimality condition is the **Euler equation**:

$$\frac{c_1}{c_0} = \left( \frac{1}{1 + \rho} (f'(k_1) + 1 - \delta) \right)^{1/\theta}$$

- We also have another optimality condition at the end:

$$k_2 = 0$$

- We finally have feasibility conditions (resource constraints)

$$k_1 = f(k_0) + (1 - \delta)k_0 - c_0$$

$$k_2 = f(k_1) + (1 - \delta)k_1 - c_1$$

as well as the given initial capital stock,  $k_0$ .

- So we have four equations in four unknowns,  $c_0, c_1, k_1, k_2$ .

# Summary of the two-period model

- These equations can be solved to find the optimal path. In particular, it gives the optimal savings rates  $s_0$  and  $s_1$  via

$$k_1 = s_0 f(k_0) + (1 - \delta)k_0$$

$$k_2 = s_1 f(k_1) + (1 - \delta)k_1$$

What's  $s_1$ ?

- Aside—some terminology to keep in mind:
  - Initial capital stock,  $k_0$ , is **a given of the model**. So capital is historically predetermined: **“state” variable**.
  - Initial consumption,  $c_0$ , is **found by solving equations**. So consumption is not predetermined: **“control” variable**.

## Next: full-fledged NGM

- The insights from the two-period model apply more generally, as we will see in the next lecture, where we will analyze the full dynamic infinite-horizon version.
- And we will ask whether decentralized market outcomes (“the invisible hand”) replicate the solution of a benevolent social planner.

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