

Lecture 8: The Neoclassical Growth Model

Part I: Planner's Problem

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Today's roadmap

- In the next few lectures we will solve the infinite-horizon **Ramsey growth problem**. We will proceed in three steps:
 1. **Planner's problem**: arrive at system of equations characterizing the solution to the optimal planning problem [today]
 2. **Competitive equilibrium**: arrive at system of equations characterizing the economy's competitive eq'm. Show that = planning allocation, so welfare th'ms again. [next time]
 3. **Eq'm analysis**: solve (as much as possible) the system of equations, work out how changes in the environment affect outcomes [main use of the NGM]
- Today we will complete the first step, and in the process connect our analysis back to the (much simpler) Solow growth model

Outline

1. Environment

Economic primitives

The planner's problem

2. Characterizing the eq'm allocation

3. Connection to the Solow Model

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The economy's primitives

- Time is discrete, $t \in \{0, 1, \dots, T\}$, where T for now is some *finite* number.
The canonical version of the NGM lets $T = \infty$; we postpone this for later. $T = 1$ gives the simple two-period model of the previous lecture note.
- **Technology**
 - There is a single good, which is produced by capital and labor, as usual. The production function $F(K, L)$ is neoclassical.
 - The good can be either consumed or invested and used as capital next period. Capital depreciates at rate $\delta \in [0, 1]$.
- **Households and preferences**
 - Population growth is fixed and L_t is normalized to 1, so aggregate and per-capita variables coincide ($K_t = k_t$, $Y_t = y_t$, $C_t = c_t$, ...).
 - All households are identical, supply their labor endowment, and value consumption.

The economy's primitives

- Since all households are identical, we can as usual simply talk about the “representative household” of the economy. Its preferences over consumption are

$$\mathcal{U}(c_0, c_1, c_2, \dots) = \sum_{t=0}^T \beta^t u(c_t) = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t u(c_t)$$

- $\beta \equiv \frac{1}{1+\rho} \in (0, 1)$ is the subjective discount factor, $\rho > 0$ is the corresponding discount rate
 - u is a strictly increasing and strictly concave function
- The output market-clearing condition is

$$y_t = f(k_t) = c_t + i_t$$

As usual $f(\bullet)$ is the intensive-form production function.

- The law of motion for the capital stock is

$$k_{t+1} = (1 - \delta)k_t + i_t$$

This environment is the same as last time. Just re-stating for emphasis and clarity.

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What will happen in this economy?

- Natural solution concept: **competitive equilibrium**
 - Households decide how much to consume, firms hire labor and capital, markets clear
 - This is what we will do next time
- Today instead: “planner’s problem”
 - Thought experiment: a benevolent social planner tells firms and households what to do to maximize household welfare
 - Why is this interesting? provides best possible outcome as reference point
Plus, as we will later see, outcomes are actually identical to competitive eq’m. This is just the First Fundamental Theorem of Welfare Economics, as already discussed many times.
 - This is also the original problem studied in Ramsey (1928): how much should a nation save?

Planner's problem: objective

- Recall the definition of Pareto Optimality—no agent can be made strictly better off without keeping other agents equally satisfied.
- With a representative agent: Pareto Optimality = maximize utility of that agent.
- Our benevolent “planner” maximizes welfare of the representative agent,

$$\mathcal{W} \equiv \sum_{t=0}^T \beta^t u(c_t)$$

What are the constraints that the planner faces?

Planner's problem: constraints

- In seeking to maximize welfare, the planner does not have to adhere to the rules of free markets—she can dictate households and firms what to do.
- The planner nevertheless faces certain **technological constraints**.

1. Only choose non-negative amounts of consumption and capital:

$$c_t \geq 0 \quad k_{t+1} \geq 0 \quad \forall t \in \{0, 1, \dots, T\}$$

2. Has to satisfy the resource constraint of the economy:

$$c_t + \underbrace{k_t - (1 - \delta)k_{t-1}}_{i_t} \leq y_t$$

where c_t denotes consumption, i_t denotes (gross) investment, and y_t denotes output (all in per capita terms). The latter is in turn given by

$$y_t = f(k_t) \equiv F(k_t, 1)$$

Planner's problem: constraints

- The planner nevertheless faces certain **technological constraints**.

3. Capital accumulates according the following law of motion:

$$k_{t+1} = (1 - \delta)k_t + i_t$$

Note that *net* investment is the change in the capital stock, $k_{t+1} - k_t$, which is equal to $i_t - \delta k_t$; i.e., net = gross minus depreciation.

- Combining everything we can restate the resource constraint in a more concise form:

$$c_t + k_{t+1} \leq (1 - \delta)k_t + f(k_t).$$

That is, current consumption plus the capital left for next period cannot exceed current total resources.

- Easy to see: will want to satisfy the resource constraint with equality. Why?

Planner's problem

Putting together the objective function as well as the constraints of the problem we arrive at the following formulation of the **planner's problem**:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^T} \mathcal{W} = \sum_{t=0}^T \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} \leq (1 - \delta)k_t + f(k_t) \quad \forall t \in \{0, 1, \dots, T\}$$

$$c_t \geq 0, \quad k_{t+1} \geq 0 \quad \forall t \in \{0, 1, \dots, T\}$$

$$k_0 > 0 \text{ given}$$

Planner's problem: technical notes

- The Theorem of the Maximum states that there always exists a solution to the problem of maximizing a continuous function over a compact set.
 - Let $\mathcal{S}$ denote the set of $\{c_t, k_{t+1}\}_{t=0}^T$ that satisfy the constraints. We call $\mathcal{S}$ the set of “feasible allocations”. Easy to verify that $\mathcal{S}$ is compact (thanks to the continuity of f) as well as convex (thanks to the concavity of f). Detailed arguments not needed for this class.
 - $\mathcal{W}$ is continuous (because u is continuous) and $\mathcal{S}$ is compact. We can therefore guarantee the *existence* of a solution to the planner's problem.
- By the strict concavity of $\mathcal{W}$ and the convexity of $\mathcal{S}$, we also have *uniqueness*.

Proposition

The solution to the planner's problem exists and is unique: there is a unique path $\{c_t, k_{t+1}\}_{t=0}^T$ that maximizes social welfare ($\mathcal{W}$) over the set of feasible allocations ($\mathcal{S}$).

Planner's problem: technical notes

- Provided that u and f are differentiable, which we assume throughout, we can then characterize the solution of the planner's problem by its first-order conditions.
The FOCs are both necessary and sufficient, given concavity of $u(\bullet)$.
- We do this next, by setting up the Lagrangian of the problem and getting the FOCs.

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Characterizing the optimal allocation

- Let μ_t denote the Lagrange multiplier for the resource constraint. The Lagrangian of the planner's problem is

$$\mathcal{L} = \sum_{t=0}^T \beta^t u(c_t) + \sum_{t=0}^T \mu_t [(1 - \delta)k_t + f(k_t) - k_{t+1} - c_t]$$

- Let $\lambda_t \equiv \beta^{-t} \mu_t$ be the multiplier in period- t terms. We can then re-write this as

$$\mathcal{L} = \sum_{t=0}^T \beta^t \{u(c_t) + \lambda_t [(1 - \delta)k_t + f(k_t) - k_{t+1} - c_t]\}$$

Note that λ_t measures the period- t shadow value of period- t resources. If you don't know this property of Lagrangeans already—we'll see it soon.

- We henceforth assume an interior solution.

We will later identify certain limit conditions of u and f —"Inada conditions"—that guarantee the solution is indeed an interior one.

Characterizing the optimal allocation

- For any $t \in \{0, \dots, T\}$, The FOC with respect to c_t gives

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Leftrightarrow u'(c_t) = \lambda_t$$

Marginal value of addt'l unit of output at date t is just $u'(c_t)$. Why? at optimum will be indifferent between eating and saving, so just eat it, giving addt'l utility $u'(c_t)$.

- For any $t \in \{0, \dots, T-1\}$, the FOC with respect to k_{t+1} gives the **Euler equation**:

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta [1 - \delta + f'(k_{t+1})] \lambda_{t+1}$$

- Finally, the FOC with respect to λ_t simply gives us back the resource constraint:

$$c_t + k_{t+1} = (1 - \delta)k_t + f(k_t)$$

Characterizing the optimal allocation

- Combining the above, we can also write the **Euler equation** as

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 - \delta + f'(k_{t+1}) \quad \forall t \in \{0, \dots, T-1\}$$

- This condition is an example of the more general efficiency conditions of a form we've seen before: marginal rate of substitution = marginal rate of transformation.
 - Here the MRS between consumption today and consumption tomorrow equals the MRT, which is simply the marginal product of capital net of depreciation (plus one).
 - = saying social marginal benefits of saving equal social marginal costs.
- We took the FOC with respect to k_{t+1} for all t up to $T-1$. What about $t = T$? That is, what about the optimal choice k_{T+1} ?
 - No interior solution here. Instead $k_{T+1} \geq 0$ has to bind, or else the planner would have left unused resources after the end of the economy.
 - Thus, instead of a FOC, we simply require that $k_{T+1} = 0$.

Characterizing the optimal allocation

Proposition

The optimal allocation, $\{c_t, k_{t+1}\}_{t=0}^T$, is the unique path that solves the following system:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 - \delta + f'(k_{t+1}), \quad (1)$$

$$k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t, \quad (2)$$

along with the following boundary conditions:

$$k_0 > 0 \text{ given, and } k_{T+1} = 0. \quad (3)$$

- Compare this with the Solow model. There we replaced (1) by

$$c_t = (1 - s)f(k_t)$$

Now we have endogenized s , given as the s_t in $c_t = (1 - s_t)f(k_t)$ consistent with (1).

Dynamics and infinite horizon

- The system (1) - (2) is a system of two difference equations in c_t and k_t .
 - In general, such a system admits multiple solutions: there are multiple paths of c_t and k_t that satisfy these two difference equations.
 - What is claimed above is that only one solution satisfies the two relevant boundary conditions, $k_0 = \text{given}$ and $k_{T+1} = 0$, and that this particular solution identifies the socially optimal plan.
- Fine as long as T is finite. But when T is infinite, the mathematics we used so far actually don't apply, because we have to deal with infinite spaces.
 - For a detailed treatment see 14.451.
 - We will present the relevant boundary condition at ∞ and its economic intuition in a couple of minutes. First one important definition.

Steady state

- Define a steady state as a fixed point to conditions (1) - (2).
 - As in the Solow model, $c = k = 0$ is a trivial steady state that I will once again ignore.
 - So let k^* and c^* denote the steady-state levels of, respectively, capital and consumption in a non-zero steady state.
- At a non-zero steady-state, the Euler condition reduces to

$$1 = \beta[1 - \delta + f'(k^*)]$$

or equivalently

$$f'(k^*) - \delta = \rho$$

- Because f' is continuous and decreasing, starting at infinity and vanishing to zero, the above condition admits a unique solution.
- The intuition of this condition is that the social return to saving (which is the MPK net of depreciation) is equated with the social cost of saving (as measured by the discount rate).

Note: we will later explore the connection between this expression and the golden rule.

Steady state

- By the resource constraint, the steady-state level of consumption is given by

$$c^* = f(k^*) - \delta k^*$$

- Combining the above, and doing the comparative statics with respect to ρ and δ , we reach the following result.

Proposition

There exists a unique (no-zero) steady state. In this steady state, the capital-labor ratio is pinned down by

$$f'(k^*) - \delta = \rho,$$

and is decreasing in both ρ and δ , where $\rho \equiv 1/\beta - 1$ is the discount rate.

Boundary condition at ∞

- As it turns out, the right boundary condition at $T = \infty$ implies that

$$\lim_{t \rightarrow \infty} k_t = k^* \quad (3')$$

- In words: ultimately you'll end up at the steady-state level of capital that we've just defined
 - Remainder of this course: when we talk about infinite-horizon version of the neoclassical growth model you should use the boundary condition (3')
- Intuition? it's the unique point at which the planner is happy to stay forever because impatience and returns to savings are perfectly balanced
 - $k_t < k^*$: returns are higher so you gradually build up your capital stock
 - $k_t > k^*$: returns are lower so you gradually deplete your capital stock

Of course no need to be able to prove this. But after the next couple of lectures you should see the intuition for these.

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Solow vs. Ramsey: steady state

In the Solow model, the saving rate was exogenously fixed. In the NGM it is endogenous. Let's first compare the two models in steady state.

- Assume Cobb-Douglas production, $f(k) = k^\alpha$, which gives $f'(k) = \alpha k^{\alpha-1} = \alpha \frac{f(k)}{k}$.
- The steady-state capital stock satisfies

$$f'(k^*) - \delta = \rho, \quad \text{or equivalently} \quad \frac{f(k^*)}{k^*} = \frac{\rho + \delta}{\alpha}$$

How does this compare to the golden rule level? When do they coincide?

- At the same time, the fact that net investment is zero in steady state means that the steady-state saving rate, s^* satisfies

$$s^* f(k^*) = \delta k^*$$

- Combining yields $s^* = \frac{\alpha \delta}{\rho + \delta}$. vs. Solow model golden rule: $s^{gold} = \alpha$.

Solow vs. Ramsey: steady state

- To repeat:

$$s^* = \frac{\alpha\delta}{\rho + \delta}$$

- So basically we have endogenized the steady-state saving rate in terms of preferences or incentives (ρ) and technology (δ, α)
 - s^* increases with α (i.e., higher when capital has higher and flatter returns) and decreases with ρ (i.e., higher when the planner is more patient). Why?
- Two more remarks:
 1. Once we move from the planner's problem to market equilibria, the analogues of ρ may capture the effects of monopoly power, other market distortions, or taxes and regulation.
 2. Note that $s_t \equiv \frac{y_t - c_t}{y_t}$ may (optimally) differ from s^* along the transition to steady state. In this sense, the mapping to the Solow model is not only to endogenize the long-run saving rate but also to allow it to be set optimally along the transition to k^* .

Solow vs. Ramsey: dynamics

Aside: can we find conditions under which the entire model—including the transition—looks like the Solow growth model?

- Yes, but they are quite special. Suppose that $u(c) = \log(c)$ and $\delta = 1$. Euler equation and market-clearing condition then become

$$\begin{aligned}\frac{1}{\beta} \frac{c_{t+1}}{c_t} &= \alpha k_{t+1}^{\alpha-1} \\ k_{t+1} &= k_t^\alpha - c_t\end{aligned}$$

- I claim that, in this model, people find it optimal to always save a fixed fraction of output—the assumption of the Solow model.
- To see this, let us simply *guess* that $c_t = (1 - s)y_t = (1 - s)k_t^\alpha$, and then verify that indeed Euler equation and market-clearing condition are consistent with that.

Solow vs. Ramsey: dynamics

- Plugging the guess into the Euler equation we get

$$\frac{1}{\beta} \frac{k_{t+1}^\alpha}{k_t^\alpha} = \alpha k_{t+1}^{\alpha-1}$$

- Output market-clearing implies that $k_{t+1} = sk_t^\alpha$. Plugging this in we get

$$\frac{1}{\beta} \frac{(sk_t^\alpha)^\alpha}{k_t^\alpha} = \alpha (sk_t^\alpha)^{\alpha-1}$$

- Cancelling terms and re-arranging:

$$s = \alpha\beta$$

- We have thus verified our guess!

Note that this works only for $T \rightarrow \infty$. For finite-horizon will fully dis-save eventually.

Solow vs. Ramsey: dynamics

- Why do we see a constant savings rate along the transition?
- Say $k_t < k^*$. In general get two effects on consumption decisions:
 1. Substitution effect: returns are higher today than in the future, so save more today
 2. Income effect: poorer today than in the future, so consume a larger share today (to smooth consumption over the lifetime)

For log utility those two effects exactly offset, giving a constant savings rate.

We will see these effects in action in the general case two lectures from now.

- Takeaway: for Cobb-Douglas production, full depreciation, and log utility, Ramsey optimal growth = Solow growth model with $s = \alpha\beta$!

Looking ahead

- The analysis in these notes falls short in two important respects:
 - We analyzed an artificial planning problem. We are, however, interested in market equilibria. Will the two be the same? When do they differ?
 - We've reduced the economy to a couple of difference equations, and we've characterized the steady state. We haven't however said much about the transition paths.
I.e., we need to try to *solve* for the transition path.
- The next two lecture notes will address these shortcomings. Payoff: can work out how changes in the economic environment (e.g., policy changes) affect savings decisions.

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