

Lecture 9: The Neoclassical Growth Model

Part II: Competitive Equilibrium

Christian Wolf

14.05, Fall 2025

This lecture

- In the previous lecture note we studied the planner's problem.
- We now turn to the competitive equilibrium of the economy.
 - Show that the competitive eq'm coincides with planner solution. Invisible hand, again.
 - As with the planner's problem, we will handle the math for the case where the economy lasts for T periods, where T can be arbitrarily large but is finite. Briefly mention $T = \infty$.
 - We will consider two different versions of the competitive equilibrium. One in which households hold capital, and one in which firms do.
Why? useful to understand better how markets deliver efficient outcomes in this economy.
- In general: competitive eq'm is what we should study (because it's a better description of actual economies). Planner's solution is the "best case".

Outline

1. Environment

Household

Firm

Equilibrium

2. Characterizing the competitive equilibrium

3. Bonus: firms own capital

Environment

- Time: discrete, indexed by $t \in \{0, 1, \dots, T\}$ where T is a finite number.
- **Technology** [just as before, repeating for completeness]
 - Single good, produced by capital and labor. Can be consumed or invested.
 - W/o investment, the existing capital stock depreciates at rate δ , where $\delta \in [0, 1]$.
- **Households**
 - Supply all their labor at wage w_t , consume, own capital and rent out to firms at rate r_t , save in either capital or riskless one-period bond with return R_t .
Bond is in zero net supply and traded among households. It's a vehicle for private borrowing and lending. Allows us to say something about financial asset returns.
 - All households are small, so they are price-takers. They are also all identical, so we can just study a representative household.

Environment

- **Firms**

- Operate a neoclassical production function, hire labor at w_t and capital at r_t on spot markets, pay profits to households
- All firms small, so they are price-takers. They are also all identical, so we can just study a representative firm.

- **Some remarks**

1. Households and firms treat prices as exogenous to their decisions, even though their actions ultimately affect prices. Key as'n is that they are small.
 - Rests on there being *many* firms and households. In particular ignore firm market power.
 - Having many but identical firms/households means we ignore any role for inequality.

Not because these issues are not important or relevant. We simply opt to abstract from them for the time being, in order to shed light on a different set of forces.
2. Bond will seem superfluous, because rep. household will have 0 holdings of it. But it's useful to think through how financial assets are priced. Fundamental approach to asset pricing.

Outline

1. Environment

Household

Firm

Equilibrium

2. Characterizing the competitive equilibrium

3. Bonus: firms own capital

The household's problem

- Let c_t denote the consumption of the representative household. Next, let k_t be the capital stock owned by the household at the beginning of period t and let b_t be its corresponding position in the bond market.
- The period- t budget constraint of the household is

$$c_t + k_{t+1} + b_{t+1} = (1 - \delta + r_t)k_t + (1 + R_t)b_t + w_t n_t + \pi_t$$

- r_t is capital rental rate, R_t is the bond interest rate, w_t is the wage rate
 - π_t is the profit (if any) received from the firms, n_t is labor supply
- The preferences of the household are

$$\sum_{t=0}^T \beta^t u(c_t)$$

- $\beta \equiv \frac{1}{1+\rho} \in (0, 1)$ is the discount factor, $\rho > 0$ is the discount rate
 - u is a strictly increasing and strictly concave function

The household's problem

- Combining the above, we can express the household's problem as:

$$\max_{\{c_t, n_t, k_{t+1}, b_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t) \quad (1)$$

subject to

$$c_t + k_{t+1} + b_{t+1} = (1 - \delta + r_t)k_t + (1 + R_t)b_t + w_t n_t + \pi_t \quad \forall t \geq 0, \quad (2)$$

$$n_t \in [0, 1], \quad c_t \geq 0, \quad k_{t+1} \geq 0, \quad b_{t+1} \geq \underline{b} \quad \forall t \geq 0, \quad (3)$$

$$k_0 > 0 \text{ given}, \quad b_0 \text{ given}, \quad b_{T+1} \geq 0 \quad (4)$$

- $\underline{b} \leq 0$ is an ad hoc, and arbitrary, limit on how much the household can borrow in every period other than the last one.
 - In the last period the limit is 0 because the household cannot die in debt.
- Choice variable is the sequence $\{c_t, n_t, k_{t+1}, b_{t+1}\}_{t=0}^T$. We say that such a sequence, or plan, is optimal for the household if it attains the maximum in the above problem.

Outline

1. Environment

Household

Firm

Equilibrium

2. Characterizing the competitive equilibrium

3. Bonus: firms own capital

The representative firm

- The rep. firm produces the single good out of capital and labor. Its output is given by

$$y_t = F(k_t^f, n_t^f)$$

where k_t^f and n_t^f denote the capital input and the labor input of the firm.

- In eq'm, k_t^f and n_t^f have to coincide with the capital stock and the labor supply of the household, so we can ultimately drop the index f .
 - We momentarily allow for the more flexible notation in order to highlight that these objects are not the same *off equilibrium*.
- The firm maximizes profits each period taking the wage rate and the rental rate as given. That is, the firm's problem at period t is

$$\pi_t \equiv \max_{\{k_t^f, n_t^f\}} [F(k_t^f, n_t^f) - r_t k_t^f - w_t n_t^f]$$

- The sequence, or plan, $\{k_t^f, n_t^f\}_{t=0}^T$ is optimal for the firm if it solves the above problem.

Outline

1. Environment

Household

Firm

Equilibrium

2. Characterizing the competitive equilibrium

3. Bonus: firms own capital

Equilibrium: definition

We specified the market structure and the problems of households and firms. We can now define our **solution concept**.

Definition

An equilibrium is a combination of prices $\{w_t, r_t, R_t\}_{t=0}^T$, a plan $\{c_t, n_t, k_{t+1}, b_{t+1}\}_{t=0}^T$ for the household, and a plan $\{k_t^f, n_t^f, \pi_t\}_{t=0}^T$ for the firm, such that the following are true:

- (i) Given the price sequence and firm profits π_t , the household plan is optimal for households. Similarly, given the price sequence, the firm plan is optimal for firms.
- (ii) All markets clear. That is:

$$k_t^f = k_t, \quad n_t^f = n_t, \quad \text{and} \quad b_t = 0. \quad \forall t$$

Outline

1. Environment

Household

Firm

Equilibrium

2. Characterizing the competitive equilibrium

3. Bonus: firms own capital

Equilibrium characterization: household

Strategy: put together the conditions that characterize the solutions to the household & firm problems with the market-clearing conditions

- Clearly households will set $n_t = 1$, so I will be imposing this from now on.
- The Lagrangian for the household problem can be written as follows:

$$\mathcal{L} = \sum_{t=0}^T \beta^t \{u(c_t) + \lambda_t [(1 - \delta)k_t + r_t k_t + w_t + (1 + R_t)b_t + \pi_t - k_{t+1} - b_{t+1} - c_t]\}$$

Assuming an interior solution, and taking the FOCs (do it!), we obtain two conditions for optimal capital and bond holdings, for all $t \leq T - 1$:

$$u'(c_t) = \beta(1 + r_{t+1} - \delta)u'(c_{t+1}) \quad (5)$$

$$u'(c_t) = \beta(1 + R_{t+1})u'(c_{t+1}) \quad (6)$$

Each condition equates the costs and benefits to saving in the asset (capital or bonds).

Equilibrium characterization: household

- Together, these conditions impose that the interest rate on bonds equal the rental rate of capital net of depreciation:

$$R_t = r_t - \delta$$

This is essentially an arbitrage condition.

- If $R_t < r_t - \delta$, the household could make a risk-free profit by borrowing a huge amount in bonds and investing it into capital. This cannot be true in eq'm, because there would be excess demand for bonds.
 - If $R_t > r_t - \delta$, no household would invest in capital, so there would be excess demand for capital (the supply of capital from the household would be zero while the demand for capital by the firms would be positive), which again cannot be true in equilibrium.
- In what follows, we therefore impose $R_{t+1} = r_{t+1} - \delta$ and concentrate on the optimality condition for capital:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 - \delta + r_{t+1} \quad (7)$$

Equilibrium characterization: household

- Next, note that clearing the bond market implies $b_t = 0$. Using this (and the fact that $n_t = 1$), we can simplify the budget constraint of the household to the following:

$$c_t + k_{t+1} = (1 - \delta + r_t)k_t + w_t + \pi_t \quad (8)$$

- That is, it is *as if* the bonds never existed: once we have imposed the arbitrage between bonds and capital, it is without loss of generality to assume there is no trade in bonds, simply because the representative household has no reason to trade bonds with itself.
 - So why did we allow for bonds in the first place? Now we can say something about the return of a financial asset (a risk-free bond).
- Aside: can use similar arguments to price *arbitrary* financial assets, with arbitrary payoff structures. General version known as consumption-based asset pricing model (C-CAPM).

Equilibrium characterization: firm

- Let us now turn to the firm's problem. As reviewed in previous lecture notes, an interior solution to this problem imposes the following:

$$r_t = F_K(k_t^f, n_t^f) \quad \text{and} \quad w_t = F_L(k_t^f, n_t^f)$$

- In equilibrium, $k_t^f = k_t$ and $n_t^f = n_t = 1$. Letting $f(k) \equiv F(k, 1)$, and recalling that $F_L(k, 1) = f(k) - f'(k)k$, we can restate this as

$$r_t = f'(k_t) \quad \text{and} \quad w_t = f(k_t) - f'(k_t)k_t \quad (9)$$

- Plugging the expression for r_t into (7) we get

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 - \delta + f'(k_{t+1})$$

- Now consider the budget constraint. By CRS we have $\pi_t = 0$. [show this!] Replacing r_t and w_t from (9) to (8), we get the resource constraint for the economy as a whole:

$$c_t + k_{t+1} = (1 - \delta)k_t + f(k_t)$$

Equilibrium characterization: summary

Combining the above findings, we reach the following eq'm characterization:

Proposition

A sequence, or “allocation”, $\{c_t, k_{t+1}\}_{t=0}^T$ is part of an equilibrium if and only if it satisfies:

$$c_t + k_{t+1} = (1 - \delta)k_t + f(k_t) \quad \forall t \leq T \quad (10)$$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 - \delta + f'(k_{t+1}) \quad \forall t \leq T - 1 \quad (11)$$

$$k_0 \text{ given} \quad \text{and} \quad k_{T+1} = 0 \quad (12)$$

For any such sequence, the associated equilibrium prices satisfy the following:

$$r_t = f'(k_t) \quad w_t = f(k_t) - f'(k_t)k_t \quad (13)$$

$$R_t = r_t - \delta \quad (14)$$

Equilibrium characterization: discussion

- Note that this proposition has two distinct parts, one about **allocations (quantities)** and another about **prices**.
 1. Conditions (10)-(11) are necessary and sufficient for an allocation to be part of an equilibrium. Note that these conditions do *not* involve prices: they permit us to test whether a candidate allocation is part of an equilibrium without looking at prices.
 2. The second part gives us the prices that “support” an equilibrium allocation, that is, the prices that have the following property: when the representative firm and household face these prices, their optimal behavior returns the allocation under consideration.
- Note that this is exactly like what we saw in the Robinson Crusoe economy.

Equilibrium: characterization

Let us focus on conditions (10)-(12), which alone pin down any eq'm allocation.

1. Condition (10) is simply the resource constraint of the economy: it describes the allocations that are technologically feasible.
2. Condition (11) equates the MRS between consumption today and tomorrow to the corresponding MRT.
 - This should remind you of something: the planner's solution! That's where we go next.
 - In some economies, like the one we study here, the incentives of the households and the firms will be such that their joint optimality ultimately guarantee the planner's optimality.
 - In other economies, such as those that feature externalities, monopoly power, or tax distortions (i.e., actual economies), this will not be true.
3. Condition (12) takes the initial capital as given (from history) and requires that the economy ends with no capital.

Equilibrium vs. planner's problem

- Comparing the dynamic system in the competitive eq'm with the one we had obtained for the planner's problem, we can conclude that:

Corollary

The eq'm allocation coincides with the solution to the planner's problem.

- Since the planner's solution exists and is unique, we also have:

Corollary

The competitive equilibrium exists and is unique.

- Two ways to interpret/use these results:
 1. Welfare theorems hold. No need for gov't intervention.
 2. Computational & conceptual: solving a planner's problem may be an indirect, powerful method for characterizing the equilibrium of the economy.

Outline

1. Environment

Household

Firm

Equilibrium

2. Characterizing the competitive equilibrium

3. Bonus: firms own capital

A different decentralization

- Our model may still seem a bit strange: in the real world households don't actually own machines—rather, they own firms/shares, and receive payouts (dividends) from firms.
- So let's consider an alternative model in which:
 - Households save only in bonds and receive dividends from firms
We will now clearly see the important role that bond prices play—signal to firms how households view consumption today vs. tomorrow, informing investment decisions.
 - Firms accumulate capital, hire labor, and pay dividends to households

Households

- Households now face the following problem:

$$\max_{\{c_t, b_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t) \quad (15)$$

subject to

$$c_t + b_{t+1} = (1 + R_t)b_t + w_t + \pi_t \quad \forall t \geq 0, \quad (16)$$

$$c_t \geq 0, \quad b_{t+1} \geq \underline{b} \quad \forall t \geq 0, \quad (17)$$

$$b_0 \text{ given}, \quad b_{T+1} \geq 0 \quad (18)$$

where π_t denotes firm profits (which will be different from before!)

Note: I am now already imposing that $n_t = 1$. This was straightforward to see anyway.

- This problem now gives the single FOC

$$u'(c_t) = \beta(1 + R_{t+1})u'(c_{t+1})$$

Firms

- Things are now more interesting on the firm side—firms face a dynamic problem!
- How do firms discount? One natural a'n: firms must generate the risk-free return R_t .
- This gives the following problem:

$$\max_{\{k_{t+1}, n_t, \pi_t\}_{t=0}^T} \sum_{t=0}^T \left(\prod_{s=0}^t \frac{1}{1 + R_s} \right) \pi_t \quad (19)$$

subject to

$$\pi_t = F(k_t, n_t) + (1 - \delta)k_t - w_t n_t - k_{t+1} \quad \forall t \geq 0, \quad (20)$$

$$k_{t+1} \geq 0 \quad \forall t \geq 0, \quad k_0 \text{ given} \quad (21)$$

We find the following two FOCs: [verify this!]

$$\begin{aligned} F_n(k_t, n_t) &= w_t = f(k_t) - f'(k_t)k_t \\ 1 &= \frac{1}{1 + R_{t+1}} [f'(k_{t+1}) + (1 - \delta)] \end{aligned}$$

Equilibrium

- Combining the household Euler equation and the firm investment equation we get

$$u'(c_t) = \beta [f'(k_{t+1}) + (1 - \delta)] u'(c_{t+1})$$

- Plugging π_t into the household budget constraint we get

$$c_t = w_t + \pi_t = f(k_t) + (1 - \delta)k_t - k_{t+1}$$

- Thus we get the *same allocation* as before! Intuition?
 - Only decision for this economy is how much to save/consume. Optimal decision depends on household preferences (β , $u(c)$) as well as returns to saving ($f'(k)$, δ).
 - Interest rate R_t encapsulates all info about household preferences. Firms then equalize this with capital returns.
 - Illustrates general point: in “frictionless” economies, prices coordinate decisions (= invisible hand). Unrealistic, but useful benchmark.

Taking Stock

Taking stock

- We have shown that the eq'm path for capital, output and consumption coincides with the socially optimal one, and we have characterized it as the solution to a system of first-order difference equations.
- But we have not yet shown how to *use* this result.
 - We have not shown how the economy evolves over time [does it go to steady state?], or how the economy responds over time to shocks and policy changes.
 - We will complete this task in the next lecture note.
- Will also see how to modify and study the eq'm when the welfare theorems don't hold (i.e., the eq'm does not coincide with the planner's solution).

Note: the generally applicable way to solve for an equilibrium is to solve household and firm problems. The planner shortcut only works in frictionless economies.

MIT OpenCourseWare
<https://ocw.mit.edu/>

14.05 Intermediate Macroeconomics
Fall 2025

For information about citing these materials or our Terms of Use, visit: <https://ocw.mit.edu/terms>.