

Problem Set 6

(due on the day of Lecture # 10)

Problem 1 (Investment and Saving in the Decentralized Open Economy) Consider a Ramsey economy in which the utility of the representative family is given by

$$U_0 = \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

and population growth is zero.

Families choose their optimal consumption path and hence savings, while the investment decision in the decentralized setting is done by firms. Families supply labor services earning a wage w_t , and own the firms receiving profits net of investment expenses, π_t . The economy differs from the standard Ramsey economy in two ways. First we allow the family to borrow and lend freely abroad at a constant interest rate ρ . Denoting debt at time t as b_t , the flow budget constraint of the family is given by

$$\dot{b}_t = \rho b_t + c_t - w_t - \pi_t.$$

At time t the family pays interest ρb_t to the rest of the world. To obtain the change in debt (the current account deficit) we have to deduct wages and net profit (received from owning the firms) from consumption.

Second, we introduce costs to installing capital. Firms produce $f(k_t)$ units of the consumption good by investing capital and employing labor. Firms rent labor services and for simplicity assume that labor is supplied inelastically so that the labor market equilibrium implies that each firm hires one worker ("family"). Firms also own capital and they finance investment through retained earnings (firms cannot rent capital anymore because of installation costs). The change in the capital stock is now given by $\dot{k}_t = i_t$. But the amount of current consumption that has to be given up in order to obtain i_t units of capital is not only i_t itself but $i_t \left(1 + a \frac{i_t}{k_t}\right)$. Installing an additional unit of capital is associated with adjustment costs of $T\left(\frac{i_t}{k_t}\right) = a \frac{i_t}{k_t}$. Notice that these costs are increasing in i_t , so installing an additional unit of capital is more expensive if the level of investment is high (but less expensive if the capital stock k_t is higher). Total adjustment costs $a \frac{i_t^2}{k_t}$ are a convex function of i_t . Therefore, a firm's net cash flow is $\pi_t = f(k_t) - w_t - i_t \left(1 + a \frac{i_t}{k_t}\right)$. The decision problem of a representative firm at time zero is to choose the time path of investment that maximizes the present discounted value of cash flows:

$$V_0 = \int_0^{\infty} e^{-\rho t} \left[f(k_t) - w_t - i_t \left(1 + a \frac{i_t}{k_t} \right) \right] dt,$$

subject to $\dot{k}_t = i_t$.

The initial capital stock and initial net foreign assets are given by k_0 and b_0 , respectively.

1. Write down the family's problem. What would the family want to do in the absence of any constraints on borrowing from the rest of the world? Write down a No-Ponzi-Game condition.
2. Set up the Hamiltonian for the family. Let μ_t be the co-state variable associated with b_t . Derive the first order conditions and the transversality condition. (Recall that μ_t is the present-value multiplier, $\mu_t = \lambda_t e^{-\rho t}$, where λ_t is the contemporaneous multiplier. So you can alternatively set up the problem with $\lambda_t e^{-\rho t}$ as the co-state and solve it in a similar way to that done in class.)
3. Show that consumption is constant on the optimal path. Why is this the case? Suppose that the interest rate is no longer equal to the time preference rate. What happens if $r > \rho$? What happens if $r < \rho$?
4. Derive the intertemporal budget constraint of the family using its dynamic budget constraint (discounting by $e^{-\rho t}$ and integrating from time 0 to ∞). Solve for $c_t = c_0$.
5. Graph total installation costs per unit of capital (per capita), $\frac{i_t}{k_t} T\left(\frac{i_t}{k_t}\right)$ against $\frac{i_t}{k_t}$. Does $T\left(\frac{i_t}{k_t}\right)$ satisfy the conditions noted in class? That is $T(0) = 0$, $T'(\cdot) > 0$, $2T'(\cdot) + \frac{i}{k} T''(\cdot) > 0$.
6. What is the firm's objective? Set up the Hamiltonian for the firm. Let $q_t e^{-\rho t}$ be the multiplier associated with the capital accumulation equation. Derive the first order conditions and the transversality condition.
7. Derive and interpret the investment function

$$\frac{i_t}{k_t} = \frac{q_t - 1}{2a}.$$

What happens as $a \rightarrow 0$?

8. Show that the optimal paths of k_t and q_t are a solution of the dynamic system

$$\begin{aligned}\dot{k}_t &= \left(\frac{q_t - 1}{2a} \right) k_t, \\ \dot{q}_t &= \rho q_t - f'(k_t) - \frac{(q_t - 1)^2}{4a}, \\ \lim_{t \rightarrow \infty} e^{-\rho t} q_t k_t &= 0, \\ k_0 &\text{ given.}\end{aligned}$$

9. Compute the steady state (k^*, q^*) of the dynamic system of part 8. and draw the phase diagram in (k, q) -space. Sketch the optimal path of k_t and q_t if $k_0 < k^*$. Discuss. What if $k_0 > k^*$?

10. Suppose $b_0 = 0$ and $k_0 < k^*$. Let $y_t = f(k_t) - i_t \left(1 + a \frac{i_t}{k_t} \right)$ be output net of adjustment cost. Show that $c_t = \rho \int_0^\infty e^{-\rho \tau} y_\tau d\tau$ for all $t \geq 0$. Argue that $\lim_{t \rightarrow \infty} y_t = f(k^*)$, $y_t < f(k^*)$ for all $t \geq 0$ and consequently $c_t < f(k^*)$ for all $t \geq 0$. Conclude that $b^* = \lim_{t \rightarrow \infty} b_t > 0$, i.e. eventually the country runs a trade surplus to pay the interest on its debt.

Problem 2 (Taxation) This problem will revisit Summer's model seen in class and first will ask you to show that it can be derived from a q-theory approach. In the paper, $q = \frac{V}{K}$. This is known more precisely as average q . Under certain conditions (which will be discussed in more detail in recitation), average q is equal to marginal q . Now return to the equation derived in class: $\dot{q} = \rho q - f'(k) - \frac{i}{k}(q - 1)$. (Here, the notation is made comparable with that seen later when solving the optimization problem so $I(q) = \frac{\dot{k}}{k} = \frac{i}{k}$.)

1. Show that this equation is approximately equivalent to that derived in the q-theory (FOC with respect to capital) if adjustment costs are of the form $T\left(\frac{i_t}{k_t}\right) = a \frac{i_t}{k_t}$. That is, show that with adjustment costs you will obtain $\dot{q} = \rho q - f'(k) - \frac{i}{k} \frac{(q-1)}{2}$. What accounts for the difference?

2. In class we saw permanent shocks to taxes on output: $(1 - \tau)f(k)$.

(a) What is the effect of a temporary unanticipated increase in τ from time 0 to time \bar{t} ? Show the effect in the phase diagram in (k, q) space.

(b) What is the effect of a temporary increase in τ announced at time 0 to take place at time \underline{t} and last until \bar{t} ? Show the effect in the phase diagram in (k, q) space.

3. Introduce a subsidy of ϕ per unit of i invested by the firm.

- (a) How does this change the firm's net cash flow?
- (b) Write a firm's Hamiltonian and derive the FOCs.
- (c) Compute the steady state (k^*, q^*) of the dynamic system.