Problem 2.2

(a) The individual's problem is to maximize lifetime utility given by

(1) $U = \dfrac{C_1^{1-\theta}}{1-\theta} + \dfrac{1}{1+\rho}\dfrac{C_2^{1-\theta}}{1-\theta}$,

subject to the lifetime budget constraint given by

(2) $P_1 C_1 + P_2 C_2 = W$,

where W represents lifetime income.

Rearrange the budget constraint to solve for C_2 :

(3) $C_2 = W/P_2 - C_1 P_1 /P_2$.

Substitute equation (3) into equation (1):

(4) $U = \dfrac{C_1^{1-\theta}}{1-\theta} + \dfrac{1}{1+\rho}\dfrac{\left[W/P_2 - C_1 P_1 /P_2\right]^{1-\theta}}{1-\theta}$.

Now we can solve the unconstrained problem of maximizing utility, as given by equation (4), with respect to first period consumption, C_1 . The first-order condition is given by

$\partial U / \partial C_1 = C_1^{-\theta} + \left(1/1+\rho\right)C_2^{-\theta}\left(-P_1 /P_2\right) = 0 \quad \Rightarrow \quad C_1^{-\theta} = \left(1/1+\rho\right)\left(P_1 /P_2\right)C_2^{-\theta}$,

or simply

(5) $C_1 = \left(1+\rho\right)^{1/\theta}\left(P_2 /P_1\right)^{1/\theta}C_2$.

In order to solve for C_2 , substitute equation (5) into equation (3):

$C_2 = W/P_2 - \left(1+\rho\right)^{1/\theta}\left(P_2 /P_1\right)^{1/\theta}C_2\left(P_1 /P_2\right) \quad \Rightarrow \quad C_2\left[1+\left(1+\rho\right)^{1/\theta}\left(P_2 /P_1\right)^{(1-\theta)/\theta}\right] = W/P_2$,

or simply

(6) $C_2 = \dfrac{W/P_2}{\left[1+\left(1+\rho\right)^{1/\theta}\left(P_2 /P_1\right)^{(1-\theta)/\theta}\right]}$.

Finally, to get the optimal choice of C_1 , substitute equation (6) into equation (5):

(7) $C_1 = \dfrac{\left(1+\rho\right)^{1/\theta}\left(P_2 /P_1\right)^{1/\theta}\left(W/P_2\right)}{\left[1+\left(1+\rho\right)^{1/\theta}\left(P_2 /P_1\right)^{(1-\theta)/\theta}\right]}$.

(b) From equation (5), the optimal ratio of first-period to second-period consumption is

(8) $C_1 /C_2 = \left(1+\rho\right)^{1/\theta}\left(P_2 /P_1\right)^{1/\theta}$.

Taking the log of both sides of equation (8) yields

(9) $\ln\left(C_1 /C_2\right) = \left(1/\theta\right)\ln\left(1+\rho\right) + \left(1/\theta\right)\ln\left(P_2 /P_1\right)$.

The elasticity of substitution between C_1 and C_2 , defined in such a way that it is positive, is given by

$\dfrac{\partial\left(C_1 /C_2\right)}{\partial\left(P_2 /P_1\right)}\dfrac{\left(P_2 /P_1\right)}{\left(C_1 /C_2\right)} = \dfrac{\partial\left[\ln\left(C_1 /C_2\right)\right]}{\partial\left[\ln\left(P_2 /P_1\right)\right]} = \dfrac{1}{\theta}$,

where we have used equation (9) to find the derivative. Thus higher values of θ imply that the individual is less willing to substitute consumption between periods.

Problem 2.5

The household's problem is to maximize lifetime utility subject to the budget constraint. That is, its problem is to maximize

(1) $U = \int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt$,

subject to

(2) $\int_{t=0}^{\infty} e^{-rt} C(t) \frac{L(t)}{H} dt = W$,

where W denotes the household's initial wealth plus the present value of its lifetime labor income, i.e. the right-hand side of equation (2.6) in the text. Note that the real interest rate, r, is assumed to be constant.

We can use the informal method, presented in the text, for solving this type of problem. Set up the Lagrangian:

$$\mathcal{L} = \int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt + \lambda \left[W - \int_{t=0}^{\infty} e^{-rt} C(t) \frac{L(t)}{H} dt \right].$$

The first-order condition is given by

$$\frac{\partial \mathcal{L}}{\partial C(t)} = e^{-\rho t} C(t)^{-\theta} \frac{L(t)}{H} - \lambda e^{-rt} \frac{L(t)}{H} = 0.$$

Canceling the L(t)/H yields

(3) $e^{-\rho t} C(t)^{-\theta} = \lambda e^{-rt}$.

Differentiate both sides of equation (3) with respect to time:

$$e^{-\rho t} \left[-\theta C(t)^{-\theta-1} \dot{C}(t) \right] - \rho e^{-\rho t} C(t)^{-\theta} + r\lambda e^{-rt} = 0.$$

This can be rearranged to obtain

(4) $-\theta \frac{\dot{C}(t)}{C(t)} e^{-\rho t} C(t)^{-\theta} - \rho e^{-\rho t} C(t)^{-\theta} + r\lambda e^{-rt} = 0$.

Now substitute the first-order condition, equation (3), into equation (4):

$$-\theta\frac{\dot{C}(t)}{C(t)}\lambda e^{-rt} - \rho\lambda e^{-rt} + r\lambda e^{-rt} = 0.$$

Canceling the λe^{-rt} and solving for the growth rate of consumption, $\dot{C}(t)/C(t)$, yields

(5) $\dfrac{\dot{C}(t)}{C(t)} = \dfrac{r-\rho}{\theta}.$

Thus with a constant real interest rate, the growth rate of consumption is a constant. If $r > \rho$ -- that is, if the rate that the market pays to defer consumption exceeds the household's discount rate -- consumption will be rising over time. The value of θ determines the magnitude of consumption growth if r exceeds ρ. A lower value of θ -- and thus a higher value of the elasticity of substitution, $1/\theta$ -- means that consumption growth will be higher for any given difference between r and ρ.

We now need to solve for the path of $C(t)$. First, note that equation (5) can be rewritten as

(6) $\dfrac{\partial \ln C(t)}{\partial t} = \dfrac{r-\rho}{\theta}.$

Integrate equation (6) forward from time $\tau = 0$ to time $\tau = t$:

$$\ln C(t) - \ln C(0) = \left[(r-\rho)/\theta\right]\tau\Big|_{\tau=0}^{\tau=t},$$

which simplifies to

(7) $\ln\left[C(t)/C(0)\right] = \left[(r-\rho)/\theta\right]t.$

Taking the exponential function of both sides of equation (7) yields

$$C(t)/C(0) = e^{\left[(r-\rho)/\theta\right]t},$$

and thus

(8) $C(t) = C(0)\,e^{\left[(r-\rho)/\theta\right]t}.$

We can now solve for initial consumption, $C(0)$, by using the fact that it must be chosen to satisfy the household's budget constraint. Substitute equation (8) into equation (2):

$$\int_{t=0}^{\infty} e^{-rt}C(0)e^{\left[(r-\rho)/\theta\right]t}\,\frac{L(t)}{H}\,dt = W.$$

Using the fact that $L(t) = L(0)e^{nt}$ yields

(9) $\dfrac{C(0)L(0)}{H}\displaystyle\int_{t=0}^{\infty}e^{-[\rho-r+\theta(r-n)]t/\theta}\,dt = W.$

As long as $[\rho - r + \theta(r - n)]/\theta > 0$, we can solve the integral:

(10) $\displaystyle\int_{t=0}^{\infty}e^{-[\rho-r+\theta(r-n)]t/\theta}\,dt = \dfrac{\theta}{\rho - r + \theta(r-n)}.$

Substitute equation (10) into equation (9) and solve for $C(0)$:

(11) $C(0) = \dfrac{W}{L(0)/H}\left[\dfrac{(\rho-r)}{\theta} + (r-n)\right].$

Finally, to get an expression for consumption at each instant in time, substitute equation (11) into equation (8):

(12) $C(t) = e^{\left[(r-\rho)/\theta\right]t}\,\dfrac{W}{L(0)/H}\left[\dfrac{(\rho-r)}{\theta} + (r-n)\right].$

<u>Problem 2.6</u>

(a) The equation describing the dynamics of the capital stock per unit of effective labor is

(1) $\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$.

For a given k, the level of c that implies $\dot{k} = 0$ is given by c = f(k) - (n + g)k. Thus a fall in g makes the level of c consistent with $\dot{k} = 0$ higher for a given k. That is, the $\dot{k} = 0$ curve shifts up. Intuitively, a lower g makes break-even investment lower at any given k and thus allows for more resources to be devoted to consumption and still maintain a given k. Since (n + g)k falls proportionately more at higher levels of k, the $\dot{k} = 0$ curve shifts up more at higher levels of k. See the figure.

(b) The equation describing the dynamics of consumption per unit of effective labor is given by

(2) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \rho - \theta g}{\theta}$.

Thus the condition required for $\dot{c} = 0$ is given by f' (k) = ρ + θg. After the fall in g, f' (k) must be lower in order for $\dot{c} = 0$. Since f" (k) is negative this means that the k needed for $\dot{c} = 0$ therefore rises. Thus the $\dot{c} = 0$ curve shifts to the right.

(c) At the time of the change in g, the value of k, the stock of capital per unit of effective labor, is given by the history of the economy, and it cannot change discontinuously. It remains equal to the k* on the old balanced growth path.

In contrast, c, the rate at which households are consuming in units of effective labor, can jump at the time of the shock. In order for the economy to reach the new balanced growth path, c must jump at the instant of the change so that the economy is on the new saddle path.
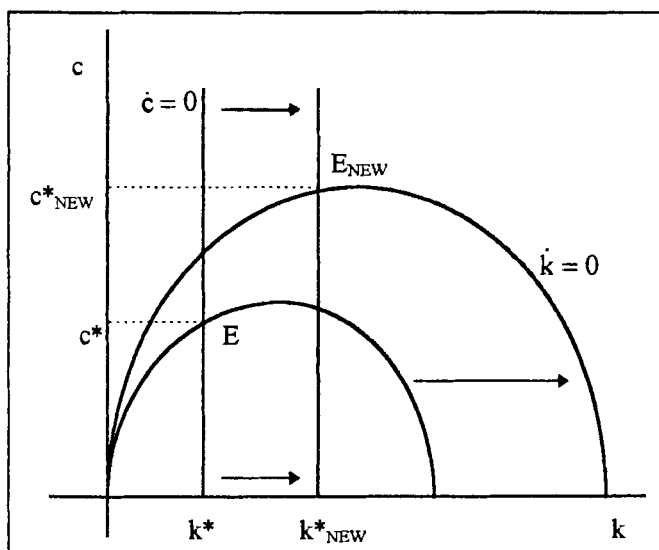
However, we cannot tell whether the new saddle path passes above or below the original point E. Thus we cannot tell whether c jumps up or down and in fact, if the new saddle path passes right through point E, c might even remain the same at the instant that g falls. Thereafter, c and k rise gradually to their new balanced-growth-path values; these are higher than their values on the original balanced growth path.

(d) On a balanced growth path, the fraction of output that is saved and invested is given by [f(k*) - c*]/f(k*). Since k is constant, or $\dot{k} = 0$ on a balanced growth path then, from equation (1), we can write f(k*) - c* = (n + g)k*. Using this, we can rewrite the fraction of output that is saved on a balanced growth path as

(3) s = [(n + g)k*]/f(k*).

Differentiating both sides of equation (3) with respect to g yields

(4) $\dfrac{\partial s}{\partial g} = \dfrac{f(k^*)[(n + g)(\partial k */\partial g) + k *] - (n + g)k * f'(k^*)(\partial k */\partial g)}{[f(k^*)]^2}$,

which simplifies to

(5) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)[f(k^*)-k^* f'(k^*)](\partial k^*/\partial g)+f(k^*)k^*}{[f(k^*)]^2}.$

Since k^* is defined by $f'(k^*) = \rho + \theta g$, differentiating both sides of this expression gives us $f''(k^*)(\partial k^*/\partial g) = \theta$. Solving for $\partial k^*/\partial g$ gives us

(6) $\partial k^*/\partial g = \theta/f''(k^*) < 0.$

Substituting equation (6) into equation (5) yields

(7) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)[f(k^*)-k^* f'(k^*)]\theta + f(k^*)k^* f''(k^*)}{[f(k^*)]^2 f''(k^*)}.$

The first term in the numerator is positive, whereas the second is negative and so the sign of $\partial s/\partial g$ is ambiguous. Thus we cannot tell whether the fall in g raises or lowers the saving rate on the new balanced growth path.

(e) When the production function is Cobb-Douglas, $f(k) = k^\alpha$, $f'(k) = \alpha k^{\alpha-1}$ and $f''(k) = \alpha(\alpha - 1)k^{\alpha-2}$. Substituting these facts into equation (7) yields

(8) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)[k^{*\alpha} - k^* \alpha k^{*\alpha-1}]\theta + k^{*\alpha} k^* \alpha(\alpha-1)k^{*\alpha-2}}{k^{*\alpha} k^{*\alpha} \alpha(\alpha-1)k^{*\alpha-2}},$

which simplifies to

(9) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)k^{*\alpha}(1-\alpha)\theta - (1-\alpha)k^{*\alpha}\alpha k^{*\alpha-1}}{[-(1-\alpha)k^{*\alpha}(\alpha k^{*\alpha-1})(\alpha k^{*\alpha-1})/\alpha]},$

which implies

(10) $\dfrac{\partial s}{\partial g} = -\alpha \dfrac{[(n+g)\theta - (\rho+\theta g)]}{(\rho+\theta g)^2}.$

Thus, finally, we have

(11) $\dfrac{\partial s}{\partial g} = -\alpha \dfrac{(n\theta-\rho)}{(\rho+\theta g)^2} = \alpha \dfrac{(\rho-n\theta)}{(\rho+\theta g)^2}.$

Problem 2.7

The two equations of motion are

(1) $\dfrac{\dot c(t)}{c(t)} = \dfrac{f'(k(t))-\rho-\theta g}{\theta},$ and (2) $\dot k(t) = f(k(t))-c(t)-(n+g)k(t).$

(a) A rise in θ or a fall in the elasticity of substitution, $1/\theta$, means that households become less willing to substitute consumption between periods. It also means that the marginal utility of consumption falls off more rapidly as consumption rises. If the economy is growing, this tends to make households value present consumption more than future consumption.

The capital-accumulation equation is unaffected. The condition required for $\dot c = 0$ is given by $f'(k) = \rho + \theta g$. Since $f''(k) < 0$, the $f'(k)$ that makes $\dot c = 0$ is now higher. Thus the value of k that satisfies $\dot c = 0$ is lower. The $\dot c = 0$ locus
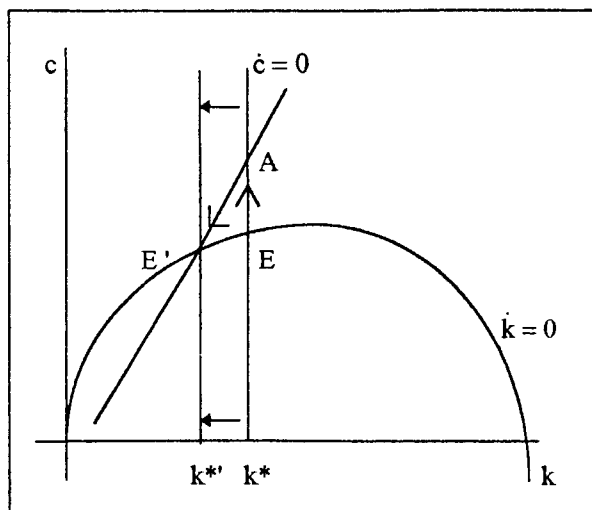
shifts to the left. The economy moves up to point A on the new saddle path; people consume more now. Movement is then down along the new saddle path until the economy reaches point E '. At that point, c* and k* are lower than their original values.

(b) We can assume that a downward shift of the production function means that for any given k, both f(k) and f'(k) are lower than before.
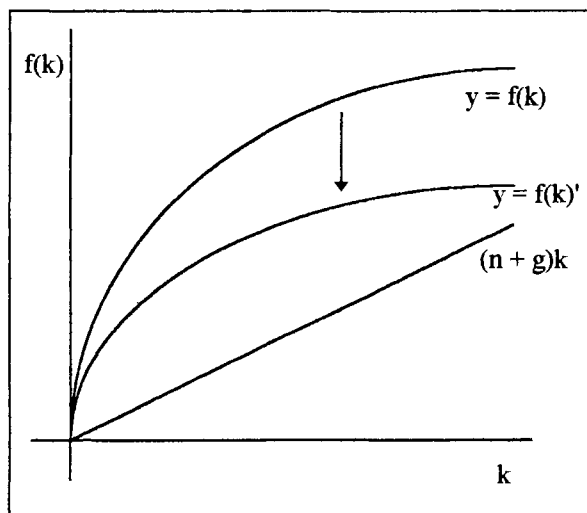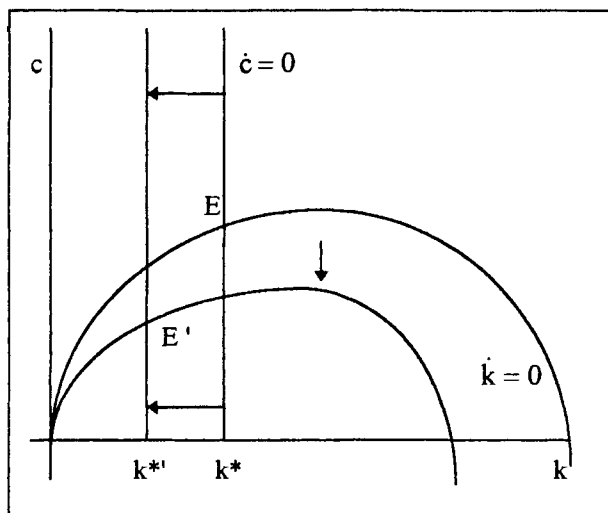

The condition required for $\dot{k} = 0$ is given by $c = f(k) - (n+g)k$. We can see from the figure on the right that the $\dot{k} = 0$ locus will shift down more at higher levels of k. Also, since for a given k, f'(k) is lower now, the golden-rule k will be lower than before. Thus the $\dot{k} = 0$ locus shifts as depicted in the figure.

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. For a given k, f'(k) is now lower. Thus we need a lower k to keep f'(k) the same and satisfy the $\dot{c} = 0$ equation. Thus the $\dot{c} = 0$ locus shifts left. The economy will eventually reach point E ' with lower c* and lower k*. Whether c initially jumps up or down depends upon whether the new saddle path passes above or below point E.

(c) With a positive rate of depreciation, $\delta > 0$, the new capital-accumulation equation is
(3) $\dot{k}(t) = f(k(t)) - c(t) - (n+g+\delta)k(t)$.
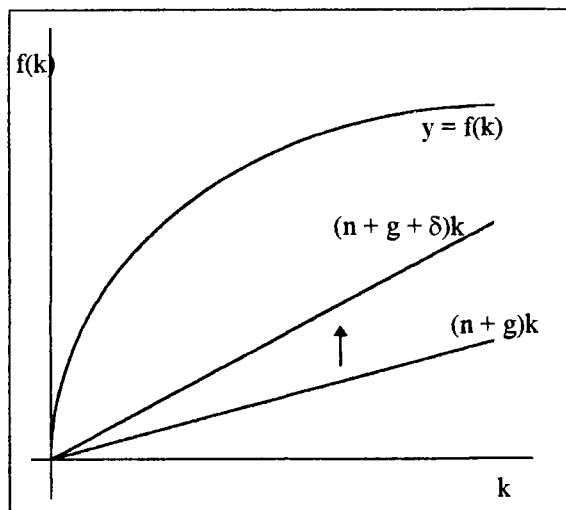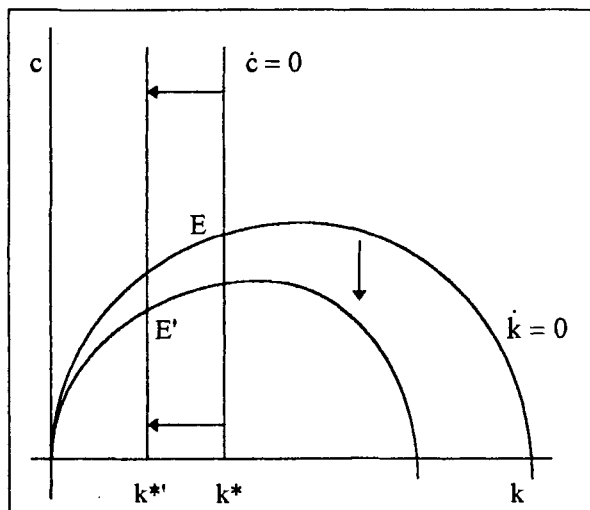
The level of saving and investment required just to keep any given k constant is now higher -- and thus the amount of consumption possible is now lower -- than in the case with no depreciation. The level of extra investment required is also higher at higher levels of k. Thus the $\dot{k} = 0$ locus shifts down more at higher levels of k.

In addition, the real return on capital is now $f'(k(t)) - \delta$ and so the household's maximization will yield

(4) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \delta - \rho - \theta g}{\theta}$.

The condition required for $\dot{c} = 0$ is $f'(k) = \delta + \rho + \theta g$. Compared to the case with no depreciation, $f'(k)$ must be higher and k lower in order for $\dot{c} = 0$. Thus the $\dot{c} = 0$ locus shifts to the left. The economy will eventually wind up at point E' with lower levels of c* and k*. Again, whether c jumps up or down initially depends upon whether the new saddle path passes above or below the original equilibrium point of E. .